Applications of Linear Algebra in Various Fields (Part-1)

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Overview

Linear algebra is used in most sciences and fields of engineering, because it allows modeling of many natural phenomena. We discuss applications of linear algebra in the following research fields.

- Computer Generating Codes
- Markov Chains
- Electric Circuits
- Linear Recurrence Relations
- Graph Theory
- Computer Graphics
- Data Fitting
- Conic Sections
- Satellite Motion

Application of Linear Algebra in Computer Generating Codes

Generating Codes with Matrices

When information is transmitted electronically, it is usual for words to be converted to numbers, which are more easily manipulated.

One can imagine many ways in which this conversion might take place, for example, by assigning to 'A' the number 1, to 'B' the number $2, \ldots$, to 'Z' the number 26, and to other characters such as punctuation marks and spaces, numbers 27 and higher.

In practice, numbers are usually expressed in 'base 2' and hence appear as sequences of 0s and 1s. Specifically, abc is the base 2 representation of the number $a(2^2) + b(2) + c(2^0)$, just as abc is the base 10 representation of the number $a(10^2) + b(10) + c(10^0)$.

For example, in base ten, 327 means $3(10^2) + 2(10) + 7(1)$. In base two, 011 means $0(2^2) + 1(2) + 1(1) = 3$.

Generating Codes with Matrices

In base two, addition and multiplication are defined "modulo 2" ("mod 2," for short), that is, by these tables

$+$	0	1
0	0	1
1	1	0

$\cdot$	0	1
0	0	0
1	0	1

The only surprise here is that $1 + 1 = 0 \pmod 2$. That is, $-1 \pmod 2$ is 1.

Generating Codes with Matrices

A word is a sequence of 0s and 1s, such as 001, and a code is a set of words. We assume that all the words of a code have the same length and that the sum of code words is also a code word. (Such a code is said to be linear.)

For example, a code might consist of all sequences of 0s and 1s of length three. There are eight such words, all are written out below:

$$000, 001, 010, 011, 100, 101, 110, 111.$$

Generating Codes with Matrices

If we identify the word *abc* with the column vector $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$, then the words of this code are just the eight possible vectors with three components, having values 0 or 1.

Equivalently, the words are linear combinations of the columns of the 3×3 identity matrix $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

In this sense, I is called a *generator matrix* for code because the words of the code are generated by (are linear combinations of) the columns.

Here is a more interesting example of a generator matrix.

Example 1 (The Hamming $(7,4)$ Code[a]).

[a] After the American Richard Hamming $(1915 - 1998)$

Let $G = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$ *and let* C *be the code whose words are linear combinations of the columns of* G*, that is, vectors of the form* $Gx, x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$.

Since $Gx + Gy = G(x + y)$*, the sum of words is a word, so* C *is indeed a (linear) code.*

The word 1010101 is in C because $\begin{bmatrix}1\\0\\1\\0\\1\\0\\1\end{bmatrix} = G\begin{bmatrix}1\\0\\1\\0\end{bmatrix}$ is the sum of the first and third columns of G (remember that we add mod 2).

On the other hand, 0101011 is not in C because it is not Gx for any x.

Generating Codes with Matrices

In real communication, over wires or through space, reliability of transmission is a problem. Words get garbled, 0s become 1s and 1s are changed to 0s. How can one be sure that the message received was the message sent?

Such concerns make the construction of codes that can detect and correct errors an important (and lucrative!) activity. A common approach in the theory of "error-correcting codes" is to append to each code word being sent a sequence of bits (a sequence of one or more 0s and 1s), the purpose of which is to make errors obvious and, ideally, to make it possible to correct errors.

Thus each transmitted word consists of a sequence of information bits, the message word itself, followed by a sequence of error correction bits.

Generating Codes with Matrices

Here is an example. Suppose the message words we wish to transmit each have length four. We might attach to each message word a fifth bit which is the sum of the first four (mod 2). Thus, if the message word is 1010, we send 10100, the final 0 being the sum $1 + 0 + 1 + 0 \pmod 2$ of the bits in the message word.

If 11111 is received, an error is immediately detected because the sum of the first four digits is $0 \neq 1 \pmod 2$. This simple attaching of a "parity check" digit makes any single error obvious, but not correctable, because the recipient has no idea which of the first four digits is wrong. Also, a single parity check digit cannot detect two errors.

Generating Codes with Matrices

Each word in the Hamming code actually consists of four information bits followed by **three** error correction bits.

For example, the actual message in 1010101 is 1010, the first four bits. The final 101 not only makes it possible to detect an error, but also to identify and correct it! We have essentially seen already how this comes about.

Generating Codes with Matrices

A word of the Hamming code corresponds to a vector of the form.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_2 + x_3 + x_4 \\ x_1 + x_3 + x_4 \\ x_1 + x_2 + x_4 \end{bmatrix},$$

so, if the fifth bit of the transmitted squence is not the sum of the second, third and fourth, or if the sixth bit is not the sum of the first, third and fourth, or if the last transmitted bit is not the sum of the first, second and fourth, the received word is not correct.

Generating Codes with Matrices

Algebraically, $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix}$ is in C if and only if

$$\begin{aligned} x_5 &= x_2 + x_3 + x_4 \\ x_6 &= x_1 + x_3 + x_4 \\ x_7 &= x_1 + x_2 + x_4, \end{aligned} \tag{1}$$

equations which can be written

$$\begin{aligned} x_2 + x_3 + x_4 + x_5 &= 0 \\ x_1 + x_3 + x_4 + x_6 &= 0 \\ x_1 + x_2 + x_4 + x_7 &= 0, \end{aligned} \tag{2}$$

using the fact that $-1 = 1 \pmod 2$.

Generating Codes with Matrices

Let $H = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$. Then $H \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} x_2 + x_3 + x_4 + x_5 \\ x_1 + x_3 + x_4 + x_6 \\ x_1 + x_2 + x_4 + x_7 \end{bmatrix}$.

So a vector $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix}$ corresponds to a code word if and only if $Hx = 0$.

The equations in (2) are called parity check equations and the matrix H a parity check matrix.

Definition 2.

If C is a code with $n \times k$ generator matrix G, a parity check matrix for C is an $(n-k) \times n$ matrix H with the property that x is in C if and only if $Hx = 0$.

For example, in the Hamming code, the generator matrix G is 7×4 $(n = 7, k = 4)$ and the parity check matrix H is 3×7.

Neither parity check equations nor a parity check matrix are unique, and some are more useful than others.

Generating Codes with Matrices

One can show that the parity check *equations* in (2) are equivalent to

$$\begin{aligned} x_4 + x_5 + x_6 + x_7 &= 0 \\ x_2 + x_3 + x_6 + x_7 &= 0 \\ x_1 + x_3 + x_5 + x_7 &= 0, \end{aligned} \tag{3}$$

in the sense that a vector $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix}$ satisfies equations (2) if and only if it satisfies equations (3).

Generating Codes with Matrices

The parity check matrix corresponding to system (3) is

$$H = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}. \tag{4}$$

Notice that the columns of H are the numbers $1, 2, 3, 4, 5, 6, 7$ written in base 2 in their natural order.

Remember that if x is really a code word, then $Hx = 0$. Since $Hx = \begin{bmatrix} x_4 + x_5 + x_6 + x_7 \\ x_2 + x_3 + x_6 + x_7 \\ x_1 + x_3 + x_5 + x_7 \end{bmatrix}$, if x_1 is transmitted incorrectly (but all other bits are correct), then $Hx = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Notice that the number 001 in base 2 is 1. If x_2 is in error (but all other bits are correct), then $Hx = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. The number 010 in base 2 is 2. If an error is made in x_3 (but all other bits are correct), then $Hx = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

The number 011 in base 2 is 3. Generally, if there is just one error in transmission, and that error is digit i, then Hx is the number i in base 2. This particular parity check matrix not only tells us whether or not a transmitted message is correct, but it also allows us to make the correction.

The $(7, 4)$ Hamming code is an example of a "single error correcting code."

Example 3.

Assuming the $(7, 4)$ *Hamming code is being used, if the word* 0010101 *is received, what was the message word?*

Since $H \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ $(\text{mod } 2)$, *the received message* 0010 *is incorrect.*

Since $001 = 1$ *in base* 2*, the error is in the first digit. The message is* 1010*.*

Exercises 4.

1. *Let* $G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ *be the generator matrix. Find a parity check matrix* H*.*

   Answer : $H = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \end{bmatrix}$ *is one possibility.*

2. *Assuming the* $(7, 4)$ *Hamming code is the code being used and the transmitted word contains at most one error, try to find the correct word assuming each of the following words is received.*

   (a) 1010101
   (b) 1011111

   Answer : We use the parity check matrix $H = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$.

   (a) *The message is* 1010.
   (b) *The message is* 1111.

Application of Linear Algebra in Markov Chains

Markov Chains

Three thousand members of the International Order of Raccoons arrive in town for a convention. They go out to dinner the night before the convention begins, three thousand strong. Fifteen hundred go to local Italian restaurants, one thousand to Chinese restaurants and five hundred go Thai.

Not everyone is pleased with their choice. The first night of the convention, for instance, of those who ate Italian the previous night, just 70% decided to eat Italian again, 20% deciding to try Chinese and 10% Thai cuisine.

The table below shows how the Raccoons changed their choices of cuisine.

		Last Night		
		I	C	T
Tonight	I	.7	.1	.1
	C	.2	.8	.2
	T	.1	.1	.7

Suppose we set $x_0 = 1500, y_0 = 1000, z_0 = 500$, the numbers eating Italian, Chinese and Thai, respectively, the night before the convention and let x_k, y_k and z_k, denote the numbers of Raccoons eating Italian, Chinese and Thai, respectively, on the kth night of the convention.

The table shows that the numbers eating Italian, Chinese and Thai on the first night of the convention are

$$x_1 = 0.7x_0 + 0.1y_0 + 0.1z_0 = 0.7(1500) + 0.1(1000) + 0.1(500) = 1200$$
$$y_1 = 0.2x_0 + 0.8y_0 + 0.2z_0 = 0.2(1500) + 0.8(1000) + 0.2(500) = 1200$$
$$z_1 = 0.1x_0 + 0.1y_0 + 0.7z_0 = 0.1(1500) + 0.1(1000) + 0.7(500) = 600.$$

Markov Chains

These numbers, $1200, 1200, 600$, are the components of Av_0 where

$$A = \begin{bmatrix} 0.7 & 0.1 & 0.1 \\ 0.2 & 0.8 & 0.2 \\ 0.1 & 0.1 & 0.7 \end{bmatrix} \text{ and } v_0 = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} = \begin{bmatrix} 1500 \\ 1000 \\ 500 \end{bmatrix} :$$

$$Av_0 = \begin{bmatrix} 0.7 & 0.1 & 0.1 \\ 0.2 & 0.8 & 0.2 \\ 0.1 & 0.1 & 0.7 \end{bmatrix} \begin{bmatrix} 1500 \\ 1000 \\ 500 \end{bmatrix} = \begin{bmatrix} 1200 \\ 1200 \\ 600 \end{bmatrix}.$$

A matrix A is called a transition *matrix* because it describes the transition of dining preferences from one night to the next. Let $v_k = \begin{bmatrix} x_k \\ y_k \\ z_k \end{bmatrix}$. Then $v_1 = Av_0$.

If the trend continues, on the second night of the convention, the numbers of Raccoons dining Italian, Chinese and Thai, respectively, will be

$$x_2 = 0.7x_1 + 0.1y_1 + 0.1z_1 = 1020$$
$$y_2 = 0.2x_1 + 0.8y_1 + 0.2z_1 = 1320$$
$$z_2 = 0.1x_1 + 0.1y_1 + 0.7z_1 = 660,$$

these being the components of $v_2 = Av_1$:

$$Av_1 = \begin{bmatrix} 0.7 & 0.1 & 0.1 \\ 0.2 & 0.8 & 0.2 \\ 0.1 & 0.1 & 0.7 \end{bmatrix} \begin{bmatrix} 1200 \\ 1200 \\ 600 \end{bmatrix} = \begin{bmatrix} 1020 \\ 1320 \\ 660 \end{bmatrix} = v_2.$$

On the third night, assuming the trend continues, the numbers will be the components of $v_3 = Av_2 = \begin{bmatrix} 912 \\ 1392 \\ 696 \end{bmatrix}$.

In the following table, we show the numbers eating Chinese, Italian and Thai over the first ten nights of the convention. The numbers are striking. Despite the overwhelming popularity of Italian cuisine at the beginning, and the relatively few people who initially preferred

	Day k					
k	0	1	2	3	4	5
x_k(Chinese)	1500	1200	1020	912	847	808
y_k (Italian)	1000	1200	1320	1392	1435	1461
z_k(Thali)	500	600	660	696	718	731

	Day k				
k	6	7	8	9	10
x_k(Chinese)	785	771	763	758	756
y_k (Italian)	1477	1786	1492	1495	1497
z_k(Thali)	738	743	746	748	749

Thai, after ten days (Raccoon conventions are long), most people are eating Italian while the numbers going Chinese and Thai are almost the same and about half the number eating Italian. We continue our study of the recurrence relation $v_{n+1} = Av_n$, but with a very special classs of matrices.

Definition 5.

A Markov matrix[a] *is a matrix* $A = [a_{ij}]$ *with each entry* $a_{ij} \geq 0$ *and the entries in each column summing to* 1*. A Markov chain is a sequence of vectors* $v_0, v_1, v_2, \ldots,$ *where* $v_{k+1} = Av_k$ *for* $k \geq 0$ *and the matrix A is Markov. The components of* v_k *are called the states of the Markov chain at step* k*.*

[a] After the Russian probabilist Andrel Markov (1856 – 1922).

In the example above, the states correspond to Italian, Chinese and Thai.

Example 6.

The matrix $\begin{bmatrix} 0.7 & 0.1 & 0.1 \\ 0.2 & 0.8 & 0.2 \\ 0.1 & 0.1 & 0.7 \end{bmatrix}$ *which appears in our restaurant example is Markov, as is* $\begin{bmatrix} 0 & \frac{2}{3} \\ 1 & \frac{1}{3} \end{bmatrix}$, *but not* $\begin{bmatrix} \frac{1}{2} & \frac{3}{4} \\ \frac{1}{2} & \frac{1}{3} \end{bmatrix}$ *and not* $\begin{bmatrix} \frac{5}{4} & \frac{1}{3} \\ -\frac{1}{4} & \frac{2}{3} \end{bmatrix}$.

Markov matrices have some remarkable properties. In order to describe some of these, we fist discuss (square) matrices each of whose rows sums to 0.

Suppose then that $B = \begin{bmatrix} b_1 & b_2 & \cdots & b_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}$ is such a matrix. Let $u = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$ be a column of n 1s.

Markov Chains

Remembering that Bx is a linear combination of the columns of B with coefficients the components of the vector x, we see that

$$Bu = 1b_1 + 1b_2 + \cdots + 1b_n = b_1 + b_2 + \cdots + b_n$$

is just the sum of the columns of B. This sum is 0 because each of the rows of B sums to 0. So $Bu = 0$; thus, B is not invertible and $\det B = 0$.

Suppose B is a square matrix each of whose **columns** sums to 0. Then each of the rows of B^T sums to 0, so $0 = \det B^T = \det B$. **Whether the rows or the columns of a matrix sum to** 0**, the determinant of the matrix must be** 0.

Definition 7.

A Markov matrix A is regular if, for some positive integer n, all entries of A^n are positive.

Let A be a Markov matrix. Then each column of $B = A - I$ sums to 0, so $\det(A - I) = 0$, showing that 1 is an eigenvalue of A. Moreover, if A is regular in the sense that some power of A has all entries positive, then the eigenspace corresponding to $\lambda = 1$ is a line, that is, it consists of scalar multiples of a single vector. (We justify this statement later in Theorem 12.)

Example 8.

The matrix $\begin{bmatrix} 0.7 & 0.1 & 0.1 \\ 0.2 & 0.8 & 0.2 \\ 0.1 & 0.1 & 0.7 \end{bmatrix}$ *is regular because it is Markov and all its entries are positive. The eigenspace corresponding to* $\lambda = 1$ *consists of multiples of* $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$. *The matrix* $A = \begin{bmatrix} 0 & \frac{2}{3} \\ 1 & \frac{1}{3} \end{bmatrix}$ *is regular because it is Markov and the entries of* $A^2 = \begin{bmatrix} \frac{2}{3} & \frac{2}{9} \\ \frac{1}{3} & \frac{7}{9} \end{bmatrix}$ *are positive. The eigenspace corresponding to* $\lambda = 1$ *consists of multiples of* $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$. *On the other hand, while* $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ *is Markov, it is not regular because the only powers of* A *are* A *and* I *so no power will have all entries positive.*

Remember that we are studying the recurrence relation $v_{n+1} = Av_n$ **with** A **Markov.** We have $v_1 = Av_0, v_2 = Av_1 = A^2v_0, v_3 = Av_2 = A^3v_0$ and, in general, $v_k = A^k v_0$ for any $k \geq 1$. To see if our guesses about the eventual dining tastes of our conventioneers are really true, we have to discover what happens to v_k, and hence to A^k, as k gets bigger and bigger.

The columns of A^k are $A^k e_1, A^k e_2, \dots, A^k e_n$ where, as always, $e_1, e_2, \dots, e_n$ denote the standard basis vectors of $\mathbb{R}^n$. It turns out that as k increases, each of these vectors $A^k e_i$ "converges" to one and the same vector v_∞ – called the *steady state* vector – and sum of the components of v_∞ is 1. (We justify this statement later in Theorem 13.)

Markov Chains

Summarizing, as k gets larger and larger, A^k converges, that is, gets closer and closer, to the matrix $B = \begin{bmatrix} v_\infty & v_\infty & \cdots & v_\infty \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}$.

For example, we shall show the powers of $A = \begin{bmatrix} 0.7 & 0.1 & 0.1 \\ 0.2 & 0.8 & 0.2 \\ 0.1 & 0.1 & 0.7 \end{bmatrix}$ converge to $B = \begin{bmatrix} 0.25 & 0.25 & 0.25 \\ 0.50 & 0.50 & 0.50 \\ 0.25 & 0.25 & 0.25 \end{bmatrix}$.

Thus, in the long run, the numbers of Raccoons eating Italian, Chinese and Thai, respectively, are the components of $Bv_0 = \begin{bmatrix} 750 \\ 1500 \\ 750 \end{bmatrix}$.

It remains only to explain how to find v_∞, this wonderful "steady state vector." This is the vector in column of B, in particular, it is the first column, $Be_1 : v_\infty = Be_1$. Remember too that B is the matrix to which A^k converges as k gets bigger and bigger. Since A^k and $A^{k+1} = A(A^k)$ both converge to B, it follows that $B = AB$,

$$\begin{array}{ccc} A^{k+1} & = & A(A^k), \\ \downarrow & & \downarrow \\ B & & B \end{array}$$

so $Av_\infty = A(Be_1) = (AB)e_1 = Be_1 = v_\infty$. Thus v_∞ is an eigenvector of A corresponding to the eigenvalue 1. Now the eigenspace of A corresponding to 1 is a line, so it consists of all multiples tv_∞ of v_∞. The sum of the components of tv_∞ is t (because the sum of the components of v_∞ is 1). There is just one vector in the eigenspace for which the sum $t = 1$, and this is v_∞. So we have a way to find this vector.

We shall see that v_∞ is the unique eigenvector corrsponding to $\lambda = 1$ whose components sum to 1.

For example, the eigenspace of $A = \begin{bmatrix} 0.7 & 0.1 & 0.1 \\ 0.2 & 0.8 & 0.2 \\ 0.1 & 0.1 & 0.7 \end{bmatrix}$ corresponding to $\lambda = 1$ is multiples of $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, so $v_\infty = t \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ for some t that we can determine from the fact that the components of v_∞ sum to 1.

The components of $t \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ sum to $4t$, so $4t = 1$, $t = \frac{1}{4}$ and $v_\infty = \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ \frac{1}{4} \end{bmatrix}$.

Example 9.

$A = \begin{bmatrix} 0 & \frac{2}{3} \\ 1 & \frac{1}{3} \end{bmatrix}$ is a regular Markov matrix. The eigenspace corresponding to $\lambda = 1$ consists of multiples of $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

The steady state vector is the only vector in this eigenspace whose components sum to 1, so it has the form $t \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ where $2t + 3t = 1$. Thus $t = \frac{1}{5}$ and the steady state vector is $v_\infty = \frac{1}{5} \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

Example 10.

Three political parties P_1, P_2, P_3 are running for office in a forthcoming election. The transition matrix

$$A = \begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \\ \frac{1}{4} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{4} & \frac{1}{3} & \frac{1}{2} \end{bmatrix}$$

shows how voter preferences are changing from week to week during the election campaign; specifically, the (i, j) entry of A shows the fraction of the voting population whose support changes from party j to party i each week.
With $j = 3$, and $i = 1, 2, 3$, for instance, we see that each week $\frac{1}{6}$ of the supporters of P_3 switch to party P_1, $\frac{1}{3}$ switch to P_2 and $\frac{1}{2}$ maintain their allegiance to party P_3.

(a) *Find the steady state vector.*

(b) *To what matrix does A^k converge as k increases?*

(c) *Assuming there are initially $3\frac{1}{2}$ million voters, one million of whom support P_1, 500,000 of whom support P_2 and two million who support P_3, how many voters will eventually support each party?*

Solution.

(a) The given matrix is Markov (columns sum to 1) and regular (all entries are positive), so there is a steady state vector which is the unique vector in the eigenspace of 1 whose components sum to 1. The eigenspace corresponding to $\lambda = 1$ consists of vectors of the form $t \begin{bmatrix} \frac{8}{9} \\ \frac{5}{6} \\ 1 \end{bmatrix}$. The sum of the components of this vector is $\frac{8}{9}t + \frac{5}{6}t + t = \frac{49}{18}t$. The vector whose components add to 1 is $v_\infty = \frac{18}{49} \begin{bmatrix} \frac{8}{9} \\ \frac{5}{6} \\ 1 \end{bmatrix} = \frac{1}{49} \begin{bmatrix} 16 \\ 15 \\ 18 \end{bmatrix}$.

(b) A^k converges to $B = \frac{1}{49} \begin{bmatrix} 16 & 16 & 16 \\ 15 & 15 & 15 \\ 18 & 18 & 18 \end{bmatrix}$, each of whose columns is v_∞.

(c) The final numbers of supporters of each party are the components of

$$B \begin{bmatrix} 1,000,000 \\ 500,000 \\ 2,000,000 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 8,000,000 \\ 7,500,000 \\ 9,000,000 \end{bmatrix} \approx 10^6 \begin{bmatrix} 1.14 \\ 1.07 \\ 1.29 \end{bmatrix},$$

so eventually, about 1.14×10^6 people will support party P_1, 1.07×10^6 will support P_2 and 1.29×10^6 will support P_3.

In the above problem, the fractions of voters eventually supporting each party are, approximately, $\frac{1.14}{3.5}$, $\frac{1.07}{3.5}$ and $\frac{1.29}{3.5}$.

These numbers, roughly, $0.326, 0.306$ and 0.369, are the components of the steady state vector: $\frac{16}{49}, \frac{15}{49}, \frac{18}{49}$. This is always the case because the vectors $v_k = A^k v_0$ approach Bv_0 as $k \to \infty$.

Now $B = \begin{bmatrix} v_\infty & v_\infty & \cdots & v_\infty \\ \downarrow & \downarrow & & \downarrow \end{bmatrix} = \begin{bmatrix} p_1 & p_1 & p_1 \\ p_2 & p_2 & p_2 \\ p_3 & p_3 & p_3 \end{bmatrix}$, so $v_k = Bv_\infty = \begin{bmatrix} p_1(x_0 + y_0 + z_0) \\ p_2(x_0 + y_0 + z_0) \\ p_3(x_0 + y_0 + z_0) \end{bmatrix}$.

Namely, if $v_0 = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$ and $v_\infty = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$, then, as $k \to \infty$, the vectors v_k approach the vector $\begin{bmatrix} p_1(x_0 + y_0 + z_0) \\ p_2(x_0 + y_0 + z_0) \\ p_3(x_0 + y_0 + z_0) \end{bmatrix}$ whose components, as fractions of the total voting population $x_0 + y_0 + z_0$, are precisely the components of v_∞.

Note that this long term behaviour is entirely independent of the initial proportions. No matter how the electorate felt initially, it will eventually support the parties in proportions that are the components of the steady state vector.

Theorem 11.

If A is a Markov matrix, then 1 is an eigenvalue of A and all other eigenvalues λ satisfy $|\lambda| \leq 1$.

Proof. We have already shown that 1 is an eigenvalue, so it remains to prove that $|\lambda| \leq 1$ for any eigenvalue λ. Our proof uses the fact that

$$|a_1 + a_2 + \cdots + a_n| \leq |a_1| + |a_2| + \cdots + |a_n|$$

for any n real numbers $a_1, a_2, a_3, \ldots, a_n$, an easy extension of the triangle equality. Moreover, if $|a_1 + a_2 + \cdots + a_n| = |a_1| + |a_2| + \cdots + |a_n|$, then the numbers a_i all have the same sign (all are nonnegative or all nonpositive). The matrices $A = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}$ and A^T have the same characteristic polynomials because $A^T - \lambda I = (A - \lambda I)^T$, so they have the same eigenvalues.

In particular, λ is an eigenvalue of $A^T = \begin{bmatrix} a_1^T & \rightarrow \\ a_2^T & \rightarrow \\ \vdots & \\ a_n^T & \rightarrow \end{bmatrix}$, so $A^T x = \lambda x$ for some nonzero vector $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$. Let $|x_k|$ be the largest of $|x_1|, |x_2|, \ldots, |x_n|$ and compare the absolute values of the k-th components of λx and $A^T x$. We have

$$\begin{aligned} |\lambda x_k| &= |a_k \cdot x| = |a_{1k}x_1 + a_{2k}x_2 + \cdots + a_{nk}x_n| \\ &\leq |a_{1k}x_1 + |a_{2k}x_2| + \cdots + |a_{nk}x_n| \quad \text{(triangle inequality)} \\ &= |a_{1k}||x_1| + |a_{2k}||x_2| + \cdots + |a_{nk}||x_n| \\ &\leq |a_{1k}||x_k| + |a_{2k}||x_k| + \cdots + |a_{nk}||x_k| \\ &= (a_{1k} + a_{2k} + \cdots + a_{nk})|x_k| = |x_k| \end{aligned} \tag{5}$$

because each $a_{ik} \geq 0$ and these numbers sum to 1. Dividing by the nonzero $|x_k|$ gives $|\lambda| \geq 1$ as desired.

The proof just given suggests another theorem.

Theorem 12.

Let λ be an eigenvalue of a regular Markov matrix A and suppose $|\lambda| = 1$. Then $\lambda = 1$ and the corresponding eigenspace is a line: it consists of scalar multiples of a single vector.

Proof. We know that if B is an $n \times n$ matrix, then (col sp $B)^{\perp}$ =null sp B^T. Here col sp B denotes the "column space" of B (the span of the columns of B) and null sp B^T denotes the "null space" of B^T, which is the set of solutions to the homogeneous system $B^T x = 0$. It follows that the dimension of null sp B^T is the dimension of (col sp $B)^{\perp}$, the orthogonal complement of col sp B, and, by a fundamental result about orthogonal complements, this is, n - dim col sp B. One of the most basic results in linear algebra says that the sum of the dimensions of the null space and column space of an $n \times n$ matrix is n, so dim null sp $B^T = n$ - dim col sp $B =$ dim null sp B.

Markov Chains

We want to prove that the eigenspace of A corresponding to $\lambda = 1$ is a line, that is, that the null space of $B = A - I$ has dimension 1, and we have just observed that this can be established by proving that dim null sp B^T = dim null sp $(A^T - I) = 1$, that is, that the eigenspace of A^T corresponding to $\lambda = 1$ is a line. We now prove the theorem.

First assume that all the entries of the Markov matrix A are positive. Let λ be an eigenvalue of A, and hence of A^T too, with $|\lambda| = 1$ and suppose $A^T x = \lambda x$. We wish to show that $\lambda = 1$ and that the eigenspace of A^T corresponding to λ is a line.

By Theorem 11, $|\lambda| \leq 1$. If $|\lambda| = 1$, then the two inequalities in (5) must be equalities. The second of these is

$$|a_{1k}||x_1| + |a_{2k}||x_2| + \cdots + |a_{nk}||x_n|$$

$$= |a_{1k}||x_k| + |a_{2k}||x_k| + \cdots + |a_{nk}||x_k|$$

Since each $a_{ik} \geq 0$, this says

$$a_{1k}(|x_k| - |x_1|) + a_{2k}(|x_k| - |x_1|) + \cdots + a_{nk}(|x_k| - |x_n|) = 0.$$

Each term on the left is nonnegative because $|x_k| \geq |x_i|$ by maximality of $|x_k|$, so the only way the sum can be 0 is if each term is 0, that is, $|x_i| = |x_k|$ for all i.

One of our preliminary remarks gives that all $a_{ik}x_i$ have the same sign. Each $a_{ik} > 0$ by assumption, so all x_i have the same sign. Since $|x_i| = |x_k|$, we get $x_i = x_k$ for all i, which says that $x = \begin{bmatrix} x_k \\ x_k \\ \vdots \\ x_k \end{bmatrix} = x_k u$, with $u = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$. This shows that the eigenspace of A^T corresponding to λ is a line, the desired result. Now u is in this eigenspace, so $A^T u = \lambda u$. But $A^T u = u$ because each row of A^T sums to 1, so $\lambda = 1$. This completes the proof in the case that the entries of A itself are positive.

For an arbitrary regular matrix, we know only that the entries of some power A^m of A are positive. Now A^m is a regular Markov matrix, so, by what we have already seen, the eigenspace of A^m corresponding to the eigenvalue 1 is a line, consisting of multiples of some vector x_0. Let λ be an eigenvalue of A with $|\lambda| = 1$ and let x be an eigenvector of A corresponding to λ. Thus $Ax = \lambda x$, which implies $A^m x = \lambda^m x$. Since $|\lambda^m| = 1$, by what we have already seen, we must have $\lambda^m = 1$. So x, being an eigenvector of A^m corresponding to $\lambda^m = 1$, is a multiple of x_0. But x was any eigenvector of A corresponding to λ. It follows that the eigenspace of A corresponding to λ is multiples of x_0.

To see that $\lambda = 1$, we note that A^{m+1} is Markov and regular. Just as we showed $\lambda^m = 1$ for the regular Markov matrix A^m, we have $\lambda^{m+1} = 1$ for the regular Markov matrix A^{m+1}. So $\lambda^{m+1} = \lambda^m, \lambda^m(\lambda - 1) = 0$, implying $\lambda = 1$ ($\lambda \neq 0$ because $|\lambda| = 1$).

In our discussion of the properties of a Markov matrix, we stated without proof that the components of the steady state vector v_∞ sum to 1.

Let $A = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}$ be a Markov matrix, let $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ and suppose that

$Ax = y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$. Then $\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = x_1 a_1 + x_2 a_2 + \cdots + x_n a_n$, so the sum of the components of y is

$$(x_1 \times \text{ sum of components of } a_1)$$

$$+(x_2 \times \text{sum of components of } a_2)$$

$$+ \cdots + (x_n \times \text{ sum of components of } a_n).$$

Markov Chains

Since the components of each a_i sum to 1, we get

$$y_1 + y_2 + \cdots + y_n = x_1 + x_2 + \cdots + x_n$$

and conclude that

$$Ax = y \quad \text{implies} \quad \text{the sum of the componetns of x is the sum of the components of y.} \qquad (6)$$

In particular, if e_i is one the standard basis vectors of $\mathbb{R}^n$, the components of e_i sum to 1, so the components of Ae_i sum to 1 as well, as do the components of $A^2e_i = A(Ae_i), A^3e_i, A^4e_i$ and, in general, A^ke_i for any positive integer k. Since A^ke_i converges to the vector v_∞, the sum of the components of v_∞ is 1 too.

The final property of a Markov matrix upon which we have relied follows.

Theorem 13.

Let A be an $n \times n$ Markov matrix and v_0 a vector. Let $v_1 = Av_0, v_2 = Av_1 = A^2 v_0, v_3 = Av_2 = A^3 v_0$, and $v_{k+1} = Av_k = A^{k+1} v_0$ in general. Then, as k grows larger and larger, the vectors v_k converge to a vector v_∞ that depends only on the sum of the components of v_0, not on v_0 itself. In particular, the vectors $A^k e_1, A^k e_2, \ldots, A^k e_n$ converge to the same vector.

Proof. We prove this only in the case that A is diagonalizable, that is, A has n linearly independent eigenvectors $x_1, x_2, \ldots, x_n$. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be corresponding eigenvalues, respectively. The matrix $P = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}$ is invertible and

$$P^{-1}AP = D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & 0 \\ \vdots & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}.$$

Thus $A = PDP^{-1}$ and $A^k = PD^kP^{-1}$ for any positive integer k. Thus $v_k = A^k v_0 = PD^kP^{-1}v_0$

$$= \begin{bmatrix} x_1 & x_2 & \cdots & x_n \\ \downarrow & \downarrow & & \downarrow \\ & & & \end{bmatrix} \begin{bmatrix} \lambda_1^k & & & \\ & \lambda_2^k & & \\ & & \ddots & \\ & & & \lambda_n^k \end{bmatrix} \underbrace{P^{-1}v_0}_{\text{t}}$$

$$= \begin{bmatrix} \lambda_1^k x_1 & \lambda_2^k x_2 & \cdots & \lambda_n^k x_n \\ \downarrow & \downarrow & & \downarrow \\ & & & \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \\ \vdots \\ t_n \end{bmatrix} = t_1\lambda_1^k x_1 + t_2\lambda_2^k x_2 + \cdots + t_n\lambda_n^k x_n,$$

where, as indicated, $t = \begin{bmatrix} t_1 \\ \vdots \\ t_n \end{bmatrix} = P^{-1}v_0$.

Summarizing,

$$v_k = t_1\lambda_1^k x_1 + t_2\lambda_2^k x_2 + \cdots + t_n\lambda_n^k x_n, \tag{7}$$

an equation that can be used to see what happens to the vectors v_k in the long run, as k increases. By Theorem 11, 1 is an eigenvalue of A and by Theorem 12, all other eigenvalues have absolute value less than 1. Order the eiqgenvalues so that $\lambda = 1$. Then $|\lambda_i| < 1$ for $i > 1$, so $\lambda_i^k \to 0$ as $k \to \infty$ and

$$v_k \to t_1 x_1. \tag{8}$$

As recorded in (6), the sum of the components of each v_k is the sum of the components of v_0 and the same is true of the limit vector $t_1 x_1$. Had we started with any of the standard basis vectors, for instance, we would have that the sum of the components of $t_1 x_1$ is 1. Let r be the sum of the components of x_1. Thus $r \neq 0$.

Suppose the staring vector v_0 has components that sum to c. It remains only to argue that the limit vector, $t_1 x_1$ depends only on c. By (6), the components of each vector v_k sum to c, hence the same is true of the limit vector, so $c = t_1 r$. Since $r \neq 0, t_1 = \frac{c}{r}$, so the vectors v_k converge to $\frac{c}{r} x_1$, a vector depending only on c and the eigenvector x_1. This completes the proof.

Example 14.

Let $A = \begin{bmatrix} \frac{1}{3} & \frac{1}{4} \\ \frac{2}{3} & \frac{3}{4} \end{bmatrix}$.

(a) *Find a matrix* P *such that* $P^{-1}AP$ *is diagonal.*

(b) *Let* $k \geq 1$ *be an integer. Find* A^k.

(c) *Use the result of part (b) to determine the matrix* B *to which* A^k *converges as* k *gets large.*

(d) *Could* B *have been determined without computing* A^k*? Explain.*

Solution :

(a) *With* $P = \begin{bmatrix} 3 & 1 \\ 8 & 1 \end{bmatrix}$, *we have* $P^{-1}AP = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{12} \end{bmatrix}$.

(b) $A^k = PD^kP^{-1} = \frac{1}{11}\begin{bmatrix} 3 & 1 \\ 8 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{12^k} \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 8 & -3 \end{bmatrix} = \begin{bmatrix} \frac{3}{11} + \frac{8}{11}\frac{1}{12^k} & \frac{3}{11} - \frac{3}{11}\frac{1}{12^k} \\ \frac{8}{11} - \frac{8}{11}\frac{1}{12^k} & \frac{8}{11} + \frac{3}{11}\frac{1}{12^k} \end{bmatrix}$.

(c) $A \to B = \begin{bmatrix} \frac{3}{11} & \frac{3}{11} \\ \frac{8}{11} & \frac{8}{11} \end{bmatrix}$.

(d) *Since* A *is a regular Markov matrix,* A *will converge to a matrix each of whose columns is* x_∞, *the unique eigenvector corresponding to* 1 *whose components sum to* 1. *The components of* $t\begin{bmatrix} 3 \\ 8 \end{bmatrix}$ *have sum* $3t + 8t = 11t$. *We want* $11t = 1$, *so* $t = \frac{1}{11}$ *and* $B = \begin{bmatrix} \frac{3}{11} & \frac{3}{11} \\ \frac{8}{11} & \frac{8}{11} \end{bmatrix}$, *as before.*

Exercise 15.

Show that $A = \begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \\ 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 0 \end{bmatrix}$ *is regular and find the steady state vector.*

Answer : $A^4 = \begin{bmatrix} \frac{1}{8} & \frac{1}{16} & \frac{1}{16} & \frac{1}{16} \\ \frac{1}{8} & \frac{3}{16} & \frac{1}{8} & \frac{1}{8} \\ \frac{1}{2} & \frac{1}{2} & \frac{9}{16} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{5}{16} \end{bmatrix}$. *All the entries are positive, so A is regular.*

The steady state vector is $\frac{4}{15} \times = \frac{1}{15} \begin{bmatrix} 1 \\ 2 \\ 8 \\ 4 \end{bmatrix}$.

Exercise 16.

After the first day of lectures one semester, one tenth of Dr. G's linear algebra students transferred to Dr. L's section and one fifth moved to Dr, P's section. After his first class, poor Dr. L lost half his students to Dr. G and quarter of then to Dr. P. Meanwhile, Dr. P was not doing so well. He found that three tenths of his class had moved to Dr. G while another tenth had moved to Dr. L. If this movement occurs lecture after lecture, determine the eventual relative proportions of students in each section.

Answer : *The eventual proportion of the original* $x_0 + y_0 + z_0$ *students in Dr. G's class is* $\frac{55}{102} \approx .539$. *The corresponding proportions for Dr. L and Dr. P are, respectively,* $\frac{12}{102} \approx .118$ *and* $\frac{35}{102} \approx .343$. *Notice that these fractions are just the components of* v_∞.

Exercise 17.

Each year the populations of British Columbia, Ontario and Newfoundland migrate as follows:

- *one quarter of those in British Columbia and one quarter of those in New foundland move to Ontario;*
- *one sixth of those in Ontario move to British Columbia and one third of those in Ontario move to Newfoundland.*

If the total population of these three provinces is ten million, what is the eventual long term distribution of the population. Does the answer depend on the initial distribution? Explain.

Answer : *The eventual distribution is* $\frac{2}{9} \approx 22\%$ *of the initial total population in British Columbia,* $\frac{1}{3} \approx 33\%$ *in Ontario and* $\frac{4}{9} \approx 44\%$ *in Newfoundland, proportions independent of the initial distribution. They are the components of the steady state vector.*

Application of Linear Algebra in Electric Circuits

Electric Circuits

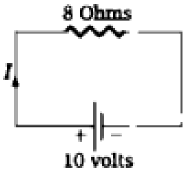
We consider electrical networks comprised of batteries and resistors connected with wires. Each battery has a positive terminal and a negative terminal causing current to flow around the circuit out of the positive terminal and into the negative terminal. We are all familiar with nine volt batteries and 1.5 volt AA batteries. "Voltage" is a measure of the power of a battery.

Each resistor, as the name implies, consumes voltage (one refers to a "voltage drop") and affects the current in the circuit. According to Ohm's Law[1] the connection between voltage E, current I (measured in amperes, amps for short) and resistance R (measured in Ohms) is given by this formula:

$$\begin{array}{ccccc} V & = & I & & R \\ volts & = & amps & \times & Ohms. \end{array}$$

[1] after Georg Ohm (1789-1854)

Electric Circuits



In a simple circuit such as the one shown above, Ohm's Law gives us two pieces of information. It says that the size of the current flowing around the circuit is $I = \frac{E}{R} = \frac{10}{8} = 1.25$ amps. It also gives us the size of the voltage drop at the resistor: $V = IR = 1.25(8) = 10$ volts.

It is standard to use the symbol  for battery and  for resistor.

A basic principle due to Gustav Kirchhoff (1824-1887) says that the sum of the voltage drops at the resistors on a circuit must equal the total voltage provided by the batteries. For the circuit on the left of the above, the voltage drops are $4I$ at one resistor and $6I$ at the other for a total of $10I$. The total voltage provided by the batteries is $3 + 12 = 15$. Kirchoff's Law says $10I = 15$, So the current around the circuit is $I = \frac{15}{10} = 1.5$ amps.

In the circuit on the left, the current flows counterclockwise around the circuit, from the positive terminal to the negative terminal in each battery. The situation is different for the circuit on the right where current leaves the positive terminal of the top battery, but enters the positive terminal of the lower battery.

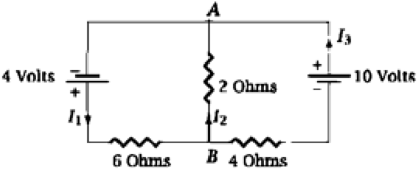
For this reason, the lower battery contributes -3 volts to the total voltage of the circuit, which is then $12 - 3 = 9$ volts. This time, we get $10I = 9$, so $I = 0.9$ amps. Since batteries contribute "signed" voltages to the total voltage in a circuit, Kirchhoff's first law is stated like this.

Kirchhoff's Circuit Law : The sum of the voltage drops around a circuit equals the algebraic sum of the voltages provided by the batteries on the circuit.

"Algebraic" means taking the direction of the current into account.

Kirchhoff's Circuit Law

The circuit depicted in the following figure contains two subcircuits which are joined at A and B. The points A and B are called nodes, these being places where one subcircuit meets another.



At A, there are two entering currents of I_2 and I_3 amps and a single departing current of I_1 amps. At B, the entering current of I_1 amps splits into currents of I_2 and I_3 amps.

Kirchhoff's Node Law

A second law is useful.

Kirchoff's Node Law : The sum of the currents flowing into any node is the sum of the currents flowing out of that node.

So from the figure,

$$\begin{aligned} At\ A: &\quad I_2 + I_3 = I_1 \\ At\ B: &\quad I_1 = I_2 + I_3. \end{aligned}$$

These equations both say $I_1 - I_2 - I_3 = 0$. In addition, Kirchhoff's Circuit Law applied to the subcircuit on the left gives

$$6I_1 + 2I_2 = 4,$$

since there are voltage drops of $6I_1$ and $2I_2$ at the two resistors on this circuit $(V = IR)$ and the total voltage supplied by the battery is 4 volts.

We must be careful with the subcircuit on the right. Do we wish to follow this circuit clockwise or counterclockwise? Suppose we follow it counterclockwise, in the direction of the arrow for I_3. Then there is a $4I_3$ voltage drop at the 4 Ohm resistor and a $-2I_2$ voltage drop at the 2 Ohm resistor because the current passing through this resistor, in the counterclockwise direction, is $-I_2$.

Kirchoff's Circuit Law gives $-2I_2 + 4I_3 = 10$. To find three currents, we must solve the system

$$\begin{aligned} I_1 - \quad I_2 \quad - I_3 &= 0 \\ 6I_1 + \quad 2I_2 \qquad &= 4 \\ -2I_2 + 4I_3 &= 10. \end{aligned} \tag{9}$$

Electric Circuits

Here is one instance of Gaussian elimination applied to the augmented matrix of coefficients:

$$\left[\begin{array}{rrr|r} 1 & -1 & -1 & 0 \\ 6 & 2 & 0 & 4 \\ 0 & -2 & 4 & 10 \end{array}\right] \to \left[\begin{array}{rrr|l} 1 & -1 & -1 & 0 \\ 0 & 8 & 6 & 4 \\ 0 & -1 & 2 & 5 \end{array}\right]$$

$$\to \left[\begin{array}{rrr|r} 1 & -1 & -1 & 0 \\ 0 & 1 & -2 & -5 \\ 0 & 4 & 3 & 2 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & -1 & -1 & 0 \\ 0 & 1 & -2 & -5 \\ 0 & 0 & 11 & 22 \end{array}\right].$$

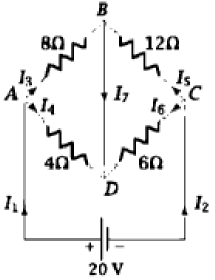
Back substitution gives $11I_3 = 22$, so $I_3 = 2$ and $I_2 - 2I_3 = -5$. We obtain $I_2 = -5 + 2I_3 = -1$ and $I_1 = I_2 + I_3 = 1$. Do not worry about the negative I_2. This simply means we put the arrow on the middle wire in the wrong direction. The current is actually flowing from A to B.

Exercise 18.

Suppose we follow the subcircuit on the right of the figure in the clockwise direction, that is, in the direction indicated by the arrow for I_2. Verify that our answer does not change.

The observant student may point out that there is a third circuit in the figure, the one that goes around the outer edge. What does Kirchhoff's Circuit Law say about this? The sum of the voltage drops is $6I_1 + 4I_3$ and the total voltage supplied by the two batteries is 14, so we get $6I_1 + 4I_3 = 14$. However, the solution remains the same.

We present in the following what is known in electrical engineering as a "Wheatstone Bridge Circuit."



If the ratio of the resistances to the left of the "bridge" BD is the same as the corresponding ratio on the right, there should be no current through the bridge. Thus, testing that $I_7 = 0$ provides a way to be sure that the ratios are the same. Let us see if this is the case.

Electric Circuits

There are four nodes, A, B, C, D where Kirchhoff's Node Law gives, respectively, these equations:

$$\begin{aligned} I_1 &= I_3 + I_4 \\ I_7 &= I_3 + I_5 \\ I_2 &= I_5 + I_6 \\ I_4 + I_6 + I_7 &= 0. \end{aligned}$$

There are three subcircuits. Kirchhoff's Law applied to the one at the bottom says $4I_4 - 6I_6 = 20$. Applied to the two triangular subcircuits at the top, the law gives $8I_3 - 4I_4 = 0$ and $6I_6 - 12I_5 = 0$.

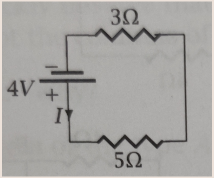
To find the individual currents, we solve the system

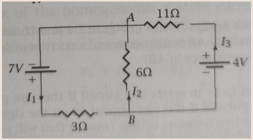
$$\begin{aligned}
I_1 - I_3 - I_4 &= 0 \\
I_3 + I_5 - I_7 &= 0 \\
I_2 - I_5 - I_6 &= 0 \\
I_4 + I_6 + I_7 &= 0 \\
4I_4 - 6I_6 &= 20 \\
8I_3 - 4I_4 &= 0 \\
-12I_5 + 6I_6 &= 0.
\end{aligned}$$

On solving, we get $I_1 = 3, I_2 = 0, I_3 = 1, I_4 = 2, I_5 = -1, I_6 = -2, I_7 = 0$. Evidence that our solution is probably correct can be obtained by checking the entire circuit, where Kirchhoff's circuit law says that $8I_3 - 12I_5 = 20$, in agreement with our results.

Example 19.

Find all indicated currents in each circuit. The standard symbol for Ohms is the Greek symbol Ω, "Omega."



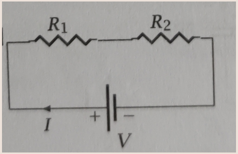


(a) $I = \frac{E}{R} = \frac{4}{8} = \frac{1}{2}$ *amps.*

(b) $I_1 = \frac{11}{9}$, $I_2 = \frac{5}{9}$, $I_3 = \frac{2}{3}$.

Exercise 20.

Two resistors are said to be in series in a circuit if they are positioned as on the left. They are in parallel if they appear as on the right. In each case, we wish to replace the two resistors by one resistor that will provide the same resistance in a simpler circuit. In each circuit, determine the "effective" single resistance of the two resistors.



Answer : *The required single resistance if* $\frac{V}{I} = \frac{V}{\frac{V}{R_1}+\frac{V}{R_2}} = \frac{1}{\frac{1}{R_1}+\frac{1}{R_2}} = \frac{R_1 R_2}{R_1+R_2}$.

Application of Linear Algebra in Linear Recurrence Relations

Here is a sequence of numbers defined "recursively:"

$$a_0 = 1, a_1 = 2 \quad \text{and,} \quad \text{for} \quad n \geq 1, \; a_{n+1} = -a_n + 2a_{n-1}. \tag{10}$$

The third equation here

$$a_{n+1} = -a_n + 2a_{n-1} \tag{11}$$

tells us how to find the terms after a_0 and a_1. To get a_2, we set $n = 1$ in (11) and find $a_2 = -a_1 + 2a_0 = 0$. We get a_3 by setting $n = 2$ in (11), and so on. Thus

$$\begin{aligned} a_3 &= -a_2 + 2a_1 = -0 + 2(2) = 4 \\ a_4 &= -a_3 + 2a_2 = -4 + 2(0) = -4 \\ a_5 &= -a_4 + 2a_3 = 4 + 2(4) = 12. \end{aligned}$$

Recursive means that each number in the sequence, after the first two, is not defined explicitly, but in terms of previous numbers in the sequence. We do not know a_{20}, for example, until we know a_{18} and a_{19}; we do not know a_{18} without knowing a_{16} and a_{17}, so it seems as if we cannot determine a_{20} until we have found all preceding terms. Actually, this is not correct. As we shall soon see, linear algebra can be used to find an explicit formula for a_n.

To begin, notice that

$$\begin{bmatrix} a_{n+1} \\ a_n \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_n \\ a_{n-1} \end{bmatrix},$$

for $n \geq 1$, and this is, $v_n = Av_{n-1}$ with $v_n = \begin{bmatrix} a_{n+1} \\ a_n \end{bmatrix}$ and $A = \begin{bmatrix} -1 & 2 \\ 1 & 0 \end{bmatrix}$.

Thus $v_1 = Av_0$, $v_2 = Av_1 = A^2 v_0$, $v_3 = Av_2 = A^3 v_0$ and, in general, $v_n = A^n v_0 = A^n \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Our job then is to find the matrix A^n, a rather unpleasant thought $A^2 = AA$, $A^3 = AAA$, $A^4 = AAAA, \dots$.

The task is much simpler, however, if we can diagonalize A. Let us use the following result from linear algebra which is one instance when we are assured that an $n \times n$ matrix has n linearly independent eigenvectors and is therefore diagonalizable.

Theorem 21.

Eigenvectors corresponding to different eigenvalues are linearly independent. Hence an $n \times n$ matrix with n different eigenvalues is diagonalizable.

The characteristic polynomial of A is

$$\det(A-\lambda I)=\begin{bmatrix}-1-\lambda & 2\\ 1 & -\lambda\end{bmatrix}=\lambda^2+\lambda-2=(\lambda+2)(\lambda-1).$$

Since the 2×2 matrix A has two distinct eigenvalues, it is diagonalizable, by Theorem 21. Thus there exists an invertible matrix P with the property that $P^{-1}AP$ is the diagonal matrix $D=\begin{bmatrix}-2 & 0\\ 0 & 1\end{bmatrix}$, whose diagonal entries are the eigenvalues of A. The columns of P are eigenvectors of A corresponding to the eigenvalues -2 and 1, respectively.

To find the eigenspace for $\lambda=-2$, we solve the homogeneous system $(A-\lambda I)x=0$ for $x=\begin{bmatrix}x_1\\ x_2\end{bmatrix}$ with $\lambda=-2$. The solution is $x=\begin{bmatrix}-2t\\ t\end{bmatrix}=t\begin{bmatrix}-2\\ 1\end{bmatrix}$.

Similarly, the eigenspace for $\lambda = 1$ consists of vectors of the form $x = \begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Placing eigenvectors in the columns (in order corresponding to $-2, 1$ respectively), a suitable $P = \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix}$, which gives $P^{-1}AP = D$. This implies $A = PDP^{-1}$. Now notice that

$$A^2 = PD^2P^{-1}, \quad A^3 = PD^3P^{-1}, \quad A^4 = PD^4P^{-1}$$

and, in general,

$$\begin{aligned} A^n = PD^nP^{-1} &= \frac{1}{3} \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} (-2)^n & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix} \\ &= \frac{1}{3} \begin{bmatrix} -(-2)^{n+1} + 1 & (-2)^{n+1} + 2 \\ -(-2)^n + 1 & (-2)^n + 2 \end{bmatrix}. \end{aligned}$$

Thus

$$v_n = \begin{bmatrix} a_{n+1} \\ a_n \end{bmatrix} = A^n \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} -(-2)^{n+1} + 1 & (-2)^{n+1} + 2 \\ -(-2)^n + 1 & (-2)^n + 2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$
$$= \frac{1}{3} \begin{bmatrix} (-2)^{n+2} + 2 + (-2)^{n+1} + 2 \\ (-2)^{n+1} + 2 + (-2)^n + 2 \end{bmatrix}.$$

The second component of this vector gives a formula for a_n:

$$a_n = \frac{1}{3}[4 - (-2)^n] = \frac{4}{3} - \frac{1}{3}(-2)^n, \quad n \geq 0.$$

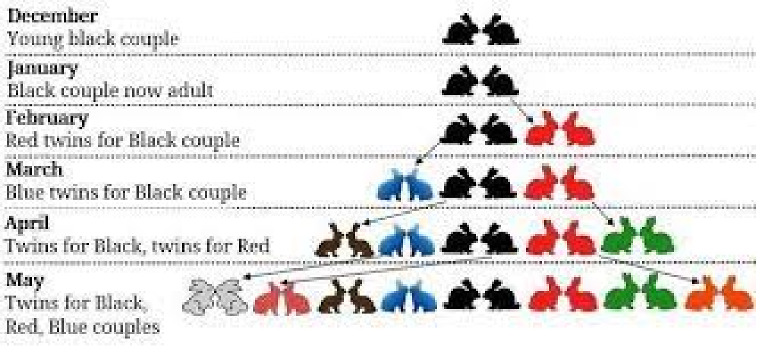
The Fibonacci Sequence

Leonardo Fibonacci (≈1180-1228), one of the brightest mathematicians of the middle ages, is credited with posing a problem about rabbits that goes more or less like this.

Suppose that newborn rabbits start producing offspring by the end of their second month of life, after which they produce a pair a month, one male and one female. Starting with one pair of rabbits on December 1, and assuming that rabbits never die, how many pairs will be alive at the end of the next year (and available for New Year's dinner)?

At the beginning, on December 1, only the initial pair is alive, and the same is true at the start of January. By February 1, however, the initial pair of rabbits has produced its first pair of offspring, so now two pairs of rabbits are alive. On March 1, the initial pair has produced one more pair and the second pair has not started reproducing yet, so three pairs are alive. On April 1, there are five pairs of rabbits. Do you see why?

The Fibonacci Sequence



At the beginning of a month, the number of pairs alive is the number alive at the start of the previous month plus the number alive two months ago (since the pairs of this last group have started to produce offspring, one pair per pair). The numbers of pairs of rabbits alive at the start of each month are the terms of the famous *Fibonacci sequence* $1, 1, 2, 3, 5, 8, \ldots$, each number after the first two being the sum of the previous two.

Linear Recurrence Relations

The Fibonacci sequence is best defined recursively like this:

$$f_0 = 1, f_1 = 1 \quad \text{and,} \quad \text{for} \;\; n \geq 1, \;\; f_{n+1} = f_n + f_{n-1},$$

which can be written

$$\begin{bmatrix} f_{n+1} \\ f_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} f_n \\ f_{n-1} \end{bmatrix}.$$

This is $v_n = Av_{n-1}$, with $v_n = \begin{bmatrix} f_{n+1} \\ f_n \end{bmatrix}$ and $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. As before, this implies $v_n = A^n v_0$ with $v_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Our job is to find A^n.

The characteristic polynomial of A is

$$det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - \lambda - 1,$$

whose roots are $\lambda_1 = \frac{1+\sqrt{5}}{2}$ and $\lambda_2 = \frac{1-\sqrt{5}}{2}$. Since the eigenvalues are different, A is diagonalizable. In fact, $P^{-1}AP = D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ with P the matrix whose columns are eigenvectors corresponding to λ_1 and λ_2, respectively. It will pay dividends now to work in some generality. Let λ be either λ_1 or λ_2 and remember that $\lambda^2 - \lambda - 1 = 0$. Thus $\lambda(\lambda - 1) = 1$, so $\lambda \neq 0$ and

$$\lambda - 1 = \frac{1}{\lambda}, \tag{12}$$

a fact that will prove very useful.

To find the eigenspace for λ, we solve the homogeneous system $(A - \lambda I)x = 0$. Gaussian elimination proceeds

$$\begin{bmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{bmatrix} \to \begin{bmatrix} 1 & \frac{1}{1-\lambda} \\ 1 & -\lambda \end{bmatrix} = \begin{bmatrix} 1 & -\lambda \\ 1 & -\lambda \end{bmatrix} \to \begin{bmatrix} 1 & -\lambda \\ 0 & 0 \end{bmatrix}.$$

with $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $x_2 = t$ is free and $x_1 = \lambda x_2 = \lambda t$, giving $x = \begin{bmatrix} \lambda t \\ t \end{bmatrix} = t \begin{bmatrix} \lambda \\ 1 \end{bmatrix}$.

It follows that $P = \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix}$, so $P^{-1} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{bmatrix}$. Now

$$\begin{bmatrix} f_{n+1} \\ f_n \end{bmatrix} = v_n = A^n v_0 = PD^n P^{-1} v_0 = PD^n P^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{5}} PD^n \begin{bmatrix} 1 - \lambda_2 \\ -1 + \lambda_1 \end{bmatrix}.$$

Since $PD^n = \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{bmatrix} = \begin{bmatrix} \lambda_1^{n+1} & \lambda_2^{n+1} \\ \lambda_1^n & \lambda_2^n \end{bmatrix}$, we have

$$\begin{aligned} \begin{bmatrix} f_{n+1} \\ f_n \end{bmatrix} &= \frac{1}{\sqrt{5}} \begin{bmatrix} \lambda_1^{n+1} & \lambda_2^{n+1} \\ \lambda_1^n & \lambda_2^n \end{bmatrix} \begin{bmatrix} 1 - \lambda_2 \\ -1 + \lambda_1 \end{bmatrix} \\ &= \frac{1}{\sqrt{5}} \left[(1 - \lambda_2) \begin{bmatrix} \lambda_1^{n+1} \\ \lambda_1^n \end{bmatrix} + (-1 + \lambda_1) \begin{bmatrix} \lambda_2^{n+1} \\ \lambda_2^n \end{bmatrix} \right]. \end{aligned}$$

Since $\lambda - 1 = \frac{1}{\lambda}$ for each λ, by (12), the second component of this vector is

$$f_n = \frac{1}{\sqrt{5}}[(1-\lambda_2)\lambda_1^n + (-1+\lambda_1)\lambda_2^n] = \frac{1}{\sqrt{5}}\left(-\frac{\lambda_1^n}{\lambda_2} + \frac{\lambda_2^n}{\lambda_1}\right)$$
$$= \frac{1}{\sqrt{5}}\left(\frac{-\lambda_1^{n+1} + \lambda_2^{n+1}}{\lambda_1\lambda_2}\right).$$

Now $\lambda_1\lambda_2 = \frac{1+\sqrt{5}}{2}\frac{1-\sqrt{5}}{2} = \frac{1-5}{4} = -1$, so $\frac{-\lambda_1^{n+1}+\lambda_2^{n+1}}{\lambda_1\lambda_2} = \lambda_1^{n+1} - \lambda_2^{n+1}$ and

$$f_n = \frac{1}{\sqrt{5}}\lambda_1^{n+1} - \frac{1}{\sqrt{5}}\lambda_2^{n+1} = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}. \tag{13}$$

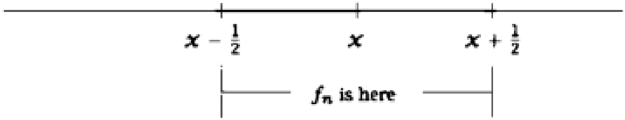
It is almost incredible that this formula should give integer values, but keeping this in mind, we can get a better formula.

Since $\left|\frac{1-\sqrt{5}}{2}\right| < 1$, $|\frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}| < \frac{1}{\sqrt{5}} < \frac{1}{2}$ and then, from (13), we discover

$$\left|f_n - \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}\right| = \left|\frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right| < \frac{1}{2}.$$

This inequality says that the integer f_n is at a distance less than $\frac{1}{2}$ from $x = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}$.

Since there is at most one such integer, we discover that f_n is the integer closest to $\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}$.



x is the integer closest to f_n

For example, since $\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{13} \approx 232.999$, we get that $f_{12} = 233$.

Exercises 22.

1. *Consider the sequence of numbers defined recursively by*

$$a_0 = 0, a_1 = 1 \textit{ and, for } n \geq 1, a_{n+1} = a_n + 2a_{n-1}.$$

   Find a general formula for a_n.

   Answer : a_n *is the second component of* $v_n = \frac{1}{3}\Big[- \begin{bmatrix} (-1)^{n+1} \\ (-1)^n \end{bmatrix} + \begin{bmatrix} 2^{n+1} \\ 2^n \end{bmatrix} \Big]$, *which is* $\frac{1}{3}[(-1)^{n+1} + 2^n]$.

2. *Let $a_0 = -4, a_1 = 0$ and, for $n \geq 1, a_{n+1} = 3a_n - 2a_{n-1}$ be a recursively defined sequence. Find a formula for a_n.*

   Answer : $a_n = -8 + 4(2^n)$.

Application of Linear Algebra in Graph Theory

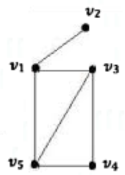
Graphs

A graph is a pair $(\mathcal{V}, E)$ of sets with $\mathcal{V}$ nonempty and each element of E a set $\{u, v\}$ of two distinct elements of $\mathcal{V}$. The elements of $\mathcal{V}$ are called vertices and the elements of E are called *edges*.

The edge $\{u, v\}$ is usually denoted just uv or vu. Edge vu is the same as uv because the sets $\{u, v\}$ and $\{v, u\}$ are the same.

One nice thing about graphs is that they can easily be pictured (if they are sufficiently small).

The following figure shows a picture of a graph G with vertex set $\mathcal{V} = \{v_1, v_2, v_3, v_4, v_5\}$ and edge set $E = \{v_1v_2, v_1v_3, v_1v_5, v_3v_4, v_4v_5, v_3v_5\}$. The five vertices are represented by solid dots, the edges by lines: there is a line between vertices v_i and v_j if v_iv_j is an edge.



Graphs arise in numerous contexts. In an obvious way, they can be used to illustrate a map of towns and connecting roads or a network of computers.

Graphs

The vertices in the graph might correspond to subjects – Math, English, Chemistry, Physics, Biology – with an edge between subjects signifying that exams in these subjects should not be scheduled at the same time. One can easily get that the three examination periods are required for the problem pictured in the figure.

In practice, graphs are often very large (lots of vertices) and it is only feasible to analyze a graph theoretical problem with the assistance of a high-speed computer. Since computers are blind, you cannot give a computer a picture; instead, it is common to give a computer the *adjacency matrix* of the graph, this being a matrix whose entries consist only of 0s and 1s, with a 1 in position (i, j) if $v_i v_j$ is an edge and otherwise a 0.

The adjacency matrix of the graph is given below.

$$A = \begin{bmatrix} 3 & 0 & 1 & 2 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 3 & 1 & 2 \\ 2 & 0 & 1 & 2 & 1 \\ 1 & 1 & 2 & 1 & 3 \end{bmatrix}.$$

The adjacency matrix of a graph is symmetric. If A is the adjacency matrix of a graph, the powers $A, A^2, A^3, \ldots$ record the number of *walks* between pairs of vertices, where a walk is a sequence of vertices with an edge between each consecutive pair. The length of a walk is the number of edges in the walk, which is one less than the number of vertices. In the graph of above figure, $v_1 v_3$ is a walk of length one from v_1 to v_3, $v_1 v_3 v_5 v_4$ is a walk of length three from v_1 to v_4, and $v_1 v_2 v_1$ is a walk of length two from v_1 to v_1.

The (i, j) entry of the adjacency matrix A is the number of walks of length **one** from v_i to v_j since there exists such a walk if and only if there is an edge between v_i and v_j. Interestingly, the (i, j) entry of A^2 is the number of walks of length **two** from v_i to v_j, and the (i, j) entry of A^3 is the number of walks of length **three** from v_i to v_j. The general situation is summarized in the next proposition.

Proposition 23.

Let A be the adjacency matrix of a graph with vertices labelled $v_1, v_2, \ldots, v_n$. For any integer $k \geq 1$, the (i, j) entry of A^k is the number walks of length k from v_i to v_j.

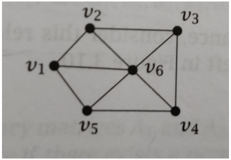
This proposition can be proved directly using a technique called "mathematical induction", so we omit the proof. The $(1, 1)$ entry of A^2 is 3, corresponding to the fact that there are three walks of length 2 from v_1 to v_1, namely, $v_1 v_2 v_1$, $v_1 v_3 v_1$ and $v_1 v_5 v_1$.

The $(3,5)$ entry is 2, corresponding to the fact that there are two walks of length 2 from v_3 to v_5, namely, $v_3v_1v_5$ and $v_3v_4v_5$.

The $(1,3)$ entry of A^3 is 6, corresponding to the six walks of length 3 from v_1 to v_3:

$$v_1v_2v_1v_3,\ v_1v_3v_1v_3,\ v_1v_5v_1v_3,\ v_1v_5v_4v_3,\ v_1v_3v_4v_3,\ v_1v_3v_5v_3.$$

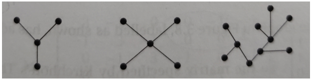
A *cycle* in a graph is a walk from a vertex v back to v that passes through vertices that are all different (except the first and the last). In the graph shown below, $v_2v_3v_6v_2$ and $v_1v_5v_4v_6v_2v_1$ are cycles, but $v_1v_6v_4v_3v_6v_1$ is not.



Spanning Trees

A *tree* is a connected graph that contains no cycles[2].

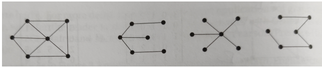
Several trees are illustrated in the following figure.



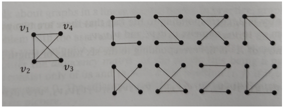
[2] "Connected" means there there is a walk from any vertex to any other vertex along a sequence of edges.

Spanning Trees

A *spanning tree* in a connected graph G is a tree that contains every vertex of G. A graph and three of its spanning trees are shown in the following figure.



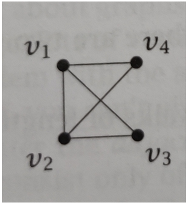
A graph and all (eight of) its spanning trees are shown below.



Now it is interesting to know how many spanning trees a given graph has. In 1847, the German physicist Gustav Kirchhoff (1824-1887) found some thing quite remarkable.

Theorem 24 (Matrix Tree Theorem / Kirchhoff's Theorem).

Let M be the matrix obtained from the adjacency matrix of a connected graph G by changing all 1s to -1s and each diagonal 0 to the degree of the corresponding vertex. Then all the cofactors of M are equal, and this common number is the number of spanning trees in G.



The graph in the last figure, has adjacency matrix $A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$, so the matrix specified by Kirchhoff's Theorem is $M = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 2 & 0 \\ -1 & -1 & 0 & 2 \end{bmatrix}$.

Using a Laplace expansion along the first row, the $(1,1)$ cofactor of M is

$$det \begin{bmatrix} 3 & -1 & -1 \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix} = 3 \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} - (-1) \begin{vmatrix} -1 & 0 \\ -1 & 2 \end{vmatrix} + (-1) \begin{vmatrix} -1 & 2 \\ -1 & 0 \end{vmatrix}$$

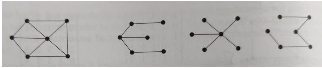
$$= 3(4) + 1(-2) + (-1)2 = 8.$$

A Laplace expansion down the third column shows that the $(2, 3)$ cofactor of M is

$$-det\begin{bmatrix} 3 & -1 & -1 \\ -1 & -1 & 0 \\ -1 & -1 & 2 \end{bmatrix} = -\left[(-1)\begin{vmatrix} -1 & -1 \\ -1 & -1 \end{vmatrix} + 2\begin{vmatrix} 3 & -1 \\ -1 & -1 \end{vmatrix}\right]$$

$$= -[(-1)(0) + 2(-3-1)] = -(-8) = 8.$$

It is no coincidence that each of these cofactors is the same. This is guaranteed by Kirchhoff's Theorem.



In the above figure, a graph G and three of its spanning trees are shown. How many spanning trees does G have in all?

Solution. With vertices labelled as shown, the adjacency matrix of G is

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}.$$

Graphs

So the matrix specified by Kirchhoff's Theorem is

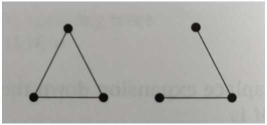
$$M = \begin{bmatrix} 3 & -1 & 0 & 0 & -1 & -1 \\ -1 & 3 & -1 & 0 & 0 & -1 \\ 0 & -1 & 3 & -1 & 0 & -1 \\ 0 & 0 & -1 & 3 & -1 & -1 \\ -1 & 0 & 0 & -1 & 3 & -1 \\ -1 & -1 & -1 & -1 & -1 & 5 \end{bmatrix}.$$

The $(1,1)$ cofactor is $\det \begin{bmatrix} 3 & -1 & 0 & 0 & -1 \\ -1 & 3 & -1 & 0 & -1 \\ 0 & -1 & 3 & -1 & -1 \\ 0 & 0 & -1 & 3 & -1 \\ 1 & -1 & -1 & -1 & 5 \end{bmatrix} = 121$, so the graph has 121 spanning trees.

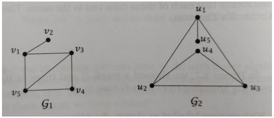
Isomorphism

The concept of "isomorphism" recurs throughout many fields of mathematics. Objects are isomorphic if they differ only in appearance rather than in some fundamental way.

In the follow figure, we show the pictures of two graphs that are fundamentally different – they have different numbers of edges.



On the other hand, the graphs given below are the same, technically, *isomorphic*.



Definition 25.

Graphs G_1 and G_2 are isomorphic if their vertices can be labelled with the same symbols in such a way that uv is an edge in G_1 if and only if uv is an edge in G_2.

For instance, consider this relabelling of vertices of the graph G_1,

$$v_1 \to u_1, \quad v_2 \to u_5, \quad v_3 \to u_2, \quad v_4 \to u_4, \quad v_5 \to u_3.$$

The edges in each graph are now the same - $u_1u_5, u_1u_2, u_1u_3, u_2u_4, u_2u_3, u_3u_4$ - so the graphs are isomorphic. Equivalently, the vertices of G_1 have been rearranged so that the adjacency matrices of the two graphs are the same. Before rearranging, G_1 and G_2 had the adjacency matrices

$$A_1 = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

respectively.

Rearranging the vertices of G_1 has the effect of rearranging the columns and the rows of A_1 in the order 15243, so that the rearranged matrix becomes A_2. Rearranging the columns of A_1 in order 15243 can be effected by multiplying this matrix on the right by the permutation matrix

$$P = \begin{bmatrix} e_1 & e_5 & e_2 & e_4 & e_3 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ & & & & \\ & & & & \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

whose columns are the standard basis vectors in order 15243 because

$$A_1 P = \begin{bmatrix} A_1 e_1 & A_1 e_5 & A_1 e_2 & A_1 e_4 & A_1 e_3 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \end{bmatrix}$$

and $A_1 e_i$ is column i of A_1.

Rearranging the rows of A_1 in the order 15243 can be accomplished by multiplying A_1 on the left by P^T, the transpose of P. To see why, write

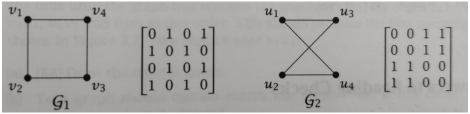
$$A_1 = \begin{bmatrix} a_1 & \rightarrow \\ a_2 & \rightarrow \\ a_3 & \rightarrow \\ a_4 & \rightarrow \\ a_5 & \rightarrow \end{bmatrix}$$ and note that

$$A_1^T P = \begin{bmatrix} A_1^T e_1 & A_1^T e_5 & A_1^T e_2 & A_1^T e_4 & A_1^T e_3 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ & & & & \end{bmatrix} = \begin{bmatrix} a_1 & a_5 & a_2 & a_4 & a_3 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ & & & & \end{bmatrix}.$$

Thus the rows of the transpose, which is $P^T A_1$, are a_1, a_5, a_2, a_4, a_3. These arguments support a general fact.

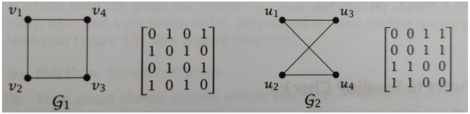
Theorem 26.

Let graphs G_1 and G_2 have adjacency matrices A_1 and A_2, respectively. Then G_1 is isomorphic to G_2 if and only if there exists a permutation matrix P with the property that $P^T A_1 P = A_2$.



The pictures of two isomorphic graphs and their adjacency matrices.

Graphs



The pictures of two isomorphic graphs, along with their adjacency matrices, are shown in the above figure. If we relabel the vertices of G_1 :

$$v_1 \to u_1, \quad v_2 \to u_4, \quad v_3 \to u_2, \quad v_4 \to u_3$$

and form the permutation matrix $P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ whose columns are the standard basis vectors in $\mathbb{R}^4$ in the order 1423, then $P^T A_1 P = A_2$.

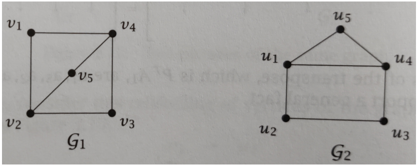
Graphs

Example 27.

Let $A_1 = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$ *and* $A_2 = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}$.

Is there a permutation matrix P *such that* $P^T A_1 P = A_2$*?*

Solution. *The matrices are the adjacency matrices of the graphs pictured in the following figure. The graphs are not isomorphic since, for example,* G_2 *contains a triangle – three vertices* u_1, u_4, u_5*, each of which is joined by an edge whereas* G_1 *has no triangles. Hence there is no permutation matrix* P *with* $P^T A_1 P = A_2$.



Exercise 28.

Consider road and rail links between four cities C_1, C_2, C_3 and C_4. Let

$$A = \begin{bmatrix} 0 & 3 & 1 & 2 \\ 3 & 0 & 1 & 0 \\ 1 & 1 & 0 & 4 \\ 2 & 0 & 4 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

where a_{ij} and b_{ij} are the number of road links and the number of rail links between C_i and C_j.
A and B are called the adjacency matrices for the road and rail networks.

Exercise 29 (contd...).

- *If the number of road links between each pair of cities is doubled show that the new adjacency matrix for the road network would be* 2A.
- *Show that* $A + B$ *is the adjacency matrix for the total network (i.e., the union of the road and rail networks).*
- *Show that* $a_{ik}a_{kj}$ *is the number of* 2*-step paths from* C_i *to* C_j *via* C_k *and that the* (i, j)*-element of* A^2 *is the number of distinct* 2*-step paths from* C_i *to* C_j*. Generalize to* A^p.
- *Show that the total number of road links at any* C_i*, which is the same as the* i*-th row sum of* A*, can be obtained as the* i*-th element of* $\mathrm{A}\mathrm{x}$ *where* x *is the* 4×1 *column matrix with all entries* 1.
- *Till now, we assumed that a link between* C_i *and* C_j *is two-way. Generalize to one-way links. Note that paths have to be properly directed now. How do you get the total number of road links from* C_i *to the other cities and the total number of road links to* C_i *from the other cities? What is the interpretation of* A^T*?*

Exercise 30.

If A is the adjacency matrix of a graph, the (i, i) entry of A^2 is the number of edges that meet at vertex v_i. Why?

Answer : *The (i, i) entry of A^2 is the number of walks of length two from v_i to v_i. Such a walk is of the form $v_i v_j v_i$ and this exists if and only if $v_i v_j$ is an edge.*

Exercise 31.

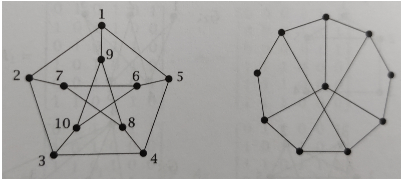
If A is the adjacency matrix of a graph, explain why the (i, j) entry of A^2 is the number of walks of length two from v_i to vj.

Answer : *The (i, j) entry of A^2 is the dot product of row i of A with column j of A. This is the number of positions k for which a_{ik} and a_{kj} are both 1. This is the number of vertices v_k for which $v_i v_k$ and $v_k v_j$ are edges and this is the number of walks of length two from v_i to v_j since any such walk is of the form $v_i v_k v_j$.*

Exercise 32.

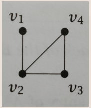
The "Petersen" graph shown on the left below is of interest in graph theory for many reasons.

1. *What is the adjency matrix of the Petersen graph, as shown?*
2. *Show that the graph on the right is isomorphic to the Petersen graph by labelling its vertices.*



Exercise 34.

Use Krichhoff's Theorem to determine the number of spanning trees in the following graph.



Answer : *The adjacency matrix is* $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$. *The matrix defined in Kirchhoff's Theorem is* $M = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 2 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix}$. *There are three spanning trees, 3 being the value of any cofactor of* M.

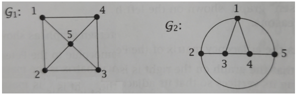
Exercise 35.

The complete graph on n *vertices is the graph with* n *vertices and all possible edges. It is denoted by* K_n. *How many spanning trees are there in* K_n*?*

Exercise 36.

Find the adjacency matrices A_1, A_2 of the graphs shown below. If there is a permutation matrix P such that $P^T A_1 P = A_2$, write down P.

Answer : *The adjacency matrices are* $A_1 = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \end{bmatrix}$ *and* $A_2 = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix}$. *The graphs are isomorphic as we see by relabelling the vertices of* G_1. *Let* $P = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$ *be the permutation matrix that has the rows of the identity in the order* 23451. *Then* $A_2 = P^T A_1 P$.



Application of Linear Algebra in Computer Graphics

Computer Graphics

We play game on computers. We watch television and we go to the movies. It is hard to escape the reality that computer graphics are all around us.

Perhaps we do not realize that linear algebra is at the heart of all the animated films of today.

We now see how matrices can be used to move and transform objects.

Such an object can be moved with the help of a matrix A by moving each point $P(x, y)$ to the point P' whose coordinates are the components of $A \begin{bmatrix} x \\ y \end{bmatrix}$, and then joining P' and Q' with a line if and only if P and Q were joined with a line in the original figure.

Remember that the columns of the matrix that effect the linear transformation T are $T(e_1)$ and $T(e_2)$.

For example, reflection in the x-axis leaves fixed $e_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, but moves $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ to the vector $\begin{bmatrix} 0 \\ -1 \end{bmatrix}$, so the matrix that performs this reflection is $\begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}$.

In general, reflection in the line with equation $y = mx$ is effected by multiplying by $\begin{bmatrix} \frac{1-m^2}{1+m^2} & \frac{2m}{m^2+1} \\ \frac{2m}{m^2+1} & \frac{m^2-1}{m^2+1} \end{bmatrix}$.

Computer Graphics

Scaling : A transformation that moves the point (x, y) to (kx, y) is called a scaling in the x-direction with factor k. If it moves (x, y) to (x, ky), it is a scaling in the y-direction with factor k.

Scaling in both the x- and y- directions by the given factors is accomplished by multiplying first by $\begin{bmatrix} k_1 & 0 \\ 0 & 1 \end{bmatrix}$ and then by $\begin{bmatrix} 1 & 0 \\ 0 & k_2 \end{bmatrix}$ (or the other way around). Since these matrices commute and the product is $\begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}$, multiplication by this one matrix accomplishes both scalings.

Shears : A shear in the x-direction with factor k moves the point (x, y) to $(x + ky, y)$.

A shear in the y-direction with factor k moves the point (x, y) to $(x, y + kx)$.

Dilations/Contractions : A transformation that moves (x, y) to (kx, ky) is a dilation if $k > 1$ and a contration if $0 < k < 1$.

Computer Graphics

Rotation through angle θ	$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$
Reflection in the x-axis	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
Reflection in the y-axis	$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$
Shear in the x-direction by factor k	$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$
Shear in the y-direction by factor k	$\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$

Computer Graphics

Reflection in $y = x$ $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Reflection in $y = mx$ $\begin{bmatrix} \frac{1-m^2}{1+m^2} & \frac{2m}{m^2+1} \\ \frac{2m}{m^2+1} & \frac{m^2-1}{m^2+1} \end{bmatrix}$

Scaling in the x-direction by factor k $\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$

Scaling in the y-direction by factor k $\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$

Dilation / contraction by k $\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$

Some Linear Transformations on $\mathbb{R}^2$

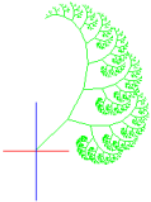


Figure 1: Basic leaf

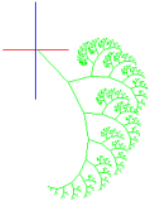


Figure 2: Reflected across x-axis

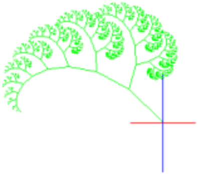


Figure 3: Rotated 90°

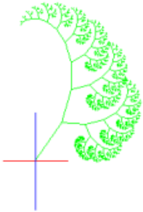


Figure 4: Rotated 10°

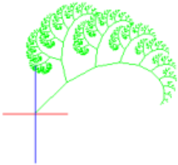


Figure 5: Reflected across the line $y = x$

Some Linear Transformations on $\mathbb{R}^2$

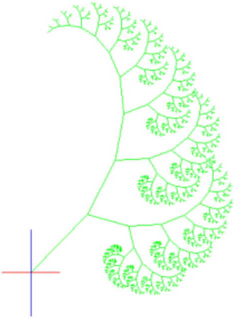


Figure 6: Expansion

Figure 7: Contraction

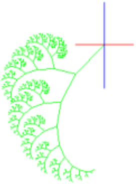


Figure 8: Half turn

Some Linear Transformations on $\mathbb{R}^2$

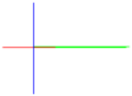


Figure 9: Projection to x-axis

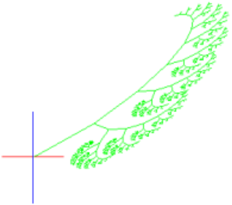


Figure 10: A shear transformation

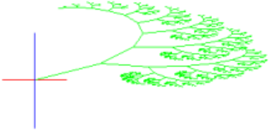


Figure 11: Stretch and squeeze

Figure 12: A rotary contraction

Homogeneous Coordinates

The one obvious transformation of objects in the plane not mentioned so far is a translation: a map of the form $(x, y) \mapsto (x + h, y + k)$ for fixed h and k. The reason, of course, is that a translation does not fix the origin, so it is not a linear transformation, hence not left multiplication by a matrix, or is it?

Look what happens if we multiply the vector $\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$ by the matrix $\begin{bmatrix} 1 & 0 & h \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix}$:

$$\begin{bmatrix} 1 & 0 & h \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x + h \\ y + k \\ 1 \end{bmatrix}.$$

Computer Graphics

So translation can be effected by changing $\begin{bmatrix} x \\ y \end{bmatrix}$ to $\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$ and multiplying by a certain 3×3 matrix. In fact, all the transformations previously discussed can be effected in a similar way, by replacing the 2×2 matrix A with the 3×3 partitioned matrix $\begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix}$ and changing the coordinates of the point $P(x, y)$ to what are called its homogeneous coordinates $(x, y, 1)$.

For example, the matrix $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ that reflects vectors in the x-axis is, in practice, replaced by the 3×3 matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, and the effect of this reflection on (x, y) is noted by observing the first two components of

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ -y \\ 1 \end{bmatrix}.$$

Similarly, the rotation matrix $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ becomes $\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and the effect of the rotation on (x, y) is seen from the first two components of

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \\ 1 \end{bmatrix}.$$

Computer Graphics

The reason for using homogeneous coordinates and converting all transformation matrices to 3×3 matrices is so that we can compose. For example, the matrix that rotates through 30° and then translates in the direction $(8, -5)$ is

$$\begin{bmatrix} 1 & 0 & 8 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\frac{\pi}{6} & -\sin\frac{\pi}{6} & 0 \\ \sin\frac{\pi}{6} & \cos\frac{\pi}{6} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} & 8 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & -5 \\ 0 & 0 & 1 \end{bmatrix}.$$

Since

$$\begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} & 8 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & -5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2}x - \frac{1}{2}y + 8 \\ \frac{1}{2}x + \frac{\sqrt{3}}{2}y - 5 \\ 1 \end{bmatrix},$$

the transformation moves (x, y) to $(\frac{\sqrt{3}2}{x} - \frac{1}{2}y + 8, \frac{1}{2}x + \frac{\sqrt{3}}{2}y - 5)$.

Application of Linear Algebra in Data Fitting

Data Fitting

We shall discuss how to find the "best solution" to a system that does not have a solution. We shall see that given a linear system $Ax = b$ that may or may not have a solution, and assuming A has linealy independent columns, then

$$x^+ = (A^T A)^{-1} A^T b$$

is the "best solution in the sense of least squares."

Suppose we want to put the "best" possible line through the three points $(-2, 4), (-1, -1), (4 - 3)$. We shall see that what we mean by "best" a little later.

Data Fitting

The line has an equation of the form $y = mx + k$. If the points in question do in face lie on a line, the coordinates (x, y) of each point would have to satisfy $y = mx + k$. So, we would have

$$\begin{aligned} 4 &= -2m + k \\ -1 &= -m + k \\ -3 &= 4m + k, \end{aligned}$$

which is $Ax = b$, with

$$A = \begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 4 \end{bmatrix}, x = \begin{pmatrix} k \\ m \end{pmatrix}, b = \begin{bmatrix} 4 \\ -1 \\ -3 \end{bmatrix}.$$

The system has no solution, of course; the points do not lie on a line. On the other hand, we can find the "best solution in the sense of least squares." We have

$$A^T A = \begin{bmatrix} 3 & 1 \\ 1 & 21 \end{bmatrix} \text{ and } (A^T A)^{-1} = \frac{1}{62}\begin{bmatrix} 21 & -1 \\ -1 & 3 \end{bmatrix}$$

so the best solution to $Ax = b$ is

$$x^+ = \begin{bmatrix} k^+ \\ m^+ \end{bmatrix} = (A^T A)^{-1} A^T b$$

$$= \frac{1}{62}\begin{bmatrix} 21 & -1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ -2 & -1 & 4 \end{bmatrix}\begin{bmatrix} 45 \\ -1 \\ -3 \end{bmatrix} = \begin{bmatrix} \frac{19}{62} \\ -\frac{57}{62} \end{bmatrix}.$$

Data Fitting

Thus $k^+ = \frac{19}{62} \approx 0.305, m^+ = -\frac{57}{62} \approx -0.919$ and our "best" line is the one with equation $y = -.919x + .305$.

In what sense is this line "best?" Remember that $x = x^+$ minimizes $\|b - Ax\|$. Now

$$b - Ax = \begin{bmatrix} 4 \\ -1 \\ -3 \end{bmatrix} - \begin{bmatrix} -2m + k \\ -m + k \\ 4m + k \end{bmatrix}$$

so $\|b - Ax\|^2$

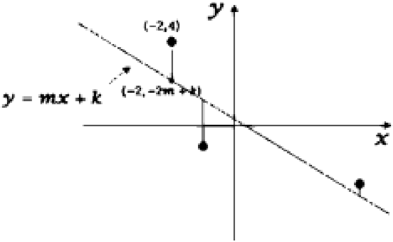
$$= [4 - (-2m + k)]^2 + [-1 - (-m + k)]^2 + [-3 - (4m + k)]^2.$$

The terms being squared here,

$$4 - (-2m + k), \quad -1 - (-m + k), \quad -3 - (4m + k)$$

are (up to sign) the deviation or vertical distances of the given points from the line.

For each pair (x_i, y_i), the vertical distance between the known data and the fit function is called the residual.



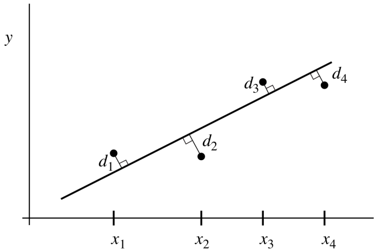
Data Fitting

The "best" line, with equation $y = -.919x + .305$, is the one for which the sum of the squares of the deviations is least.

The deviations are (approximately) $1.855, -2.226)$ and -0.371 and so the minimum sum of squares of deviations is $1.885^2 + (-2.226)^2 + 3.71^2 = 8.534$. For no other line is the corresponding sum smaller.

Orthogonal Distance Fit

An alternative to minimizing the residual is to minimize the orthogonal distance to the line.



Minimizing $\sum d_i^2$ is known as the Orthogonal Distance Regression problem. But we consider the vertical distance for deviation.

Data Fitting

We consider a more complicated situation. There may be reason to believe that only experimental error explains why the points $(-3, -.5), (-1, -.5), (1, .9), (2, 1.1), (3, 3.3)$ do not lie on a line. What line? We proceed as before. If the points were to lie on the line with equation $y = mx + k$, we would have

$$\begin{aligned} -.5 &= -3m + k \\ -.5 &= -m + k \\ .9 &= m + k \\ 1.1 &= 2m + k \\ 3.3 &= 3m + k, \end{aligned}$$

which is $Ax = b$,

with

$$A = \begin{bmatrix} 1 & -3 \\ 1 & -1 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}, x = \begin{bmatrix} k \\ m \end{bmatrix} \text{ and } b = \begin{bmatrix} -.5 \\ -.5 \\ .9 \\ 1.1 \\ 3.3 \end{bmatrix}.$$

x	y	$y^+ = m^+x + k^+$	$dev = y - y^+$	dev^2
-3	-0.5	-1.085	0.585	0.342
-1	-0.5	0.059	-0.559	0.312
1	0.9	1.203	-0.303	$.092$
2	1.1	1.775	-0.675	0.456
3	3.3	2.347	0.953	0.908
		Sum of squares of deviations	$=$	2.110

We have $A^T A = \begin{bmatrix} 5 & 2 \\ 2 & 24 \end{bmatrix}$ and $(A^T A)^{-1} = \frac{1}{116}\begin{bmatrix} 24 & -2 \\ -2 & 5 \end{bmatrix}$, so the best solution to $Ax = b$ is

$$x^+ = \begin{bmatrix} k^+ \\ m^+ \end{bmatrix} = (A^T A)^{-1} A^T b$$

$$= \frac{1}{116}\begin{bmatrix} 24 & -2 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -3 & -1 & 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} -.5 \\ -.5 \\ .9 \\ 1.1 \\ 3.3 \end{bmatrix} = \begin{bmatrix} .632 \\ .572 \end{bmatrix}.$$

Thus $k^+ = .631, m^+ = .572$, and our "best" line is the one with equation $y = .572x + .631$. The sum of squares of deviations is 2.110. This number is pretty small, suggesting that the points are, on average, pretty close to a line. The line with equation $y = .572x + .631$ is "best" in the sense that 2.110 is least: no other line has the sum of squares of the deviations of the points from the line less than 2.110.

Data Fitting

Perhaps we believe that the points listed in $(-3, -.5), (-1, -.5)$, $(1, .9)$, $(2, 1.1)$, and $(3, 3.3)$ should lie on a parabola, not a line.

If the parabola has equation $y = rx^2 + sx + t$, the coordinates of each point would satisfy this equation; that is

$$\begin{aligned} -.5 &= r(-3)^2 - 3s + t \\ -.5 &= r(-1)^2 - s + t \\ .9 &= r + s + t \\ 1.1 &= r(2^2) + 2s + t \\ 3.3 &= r(3^2) + 3s + t, \end{aligned}$$

which is $Ax = b$, with

$$A = \begin{bmatrix} 1 & -3 & 9 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix}, x = \begin{bmatrix} t \\ s \\ r \end{bmatrix}, b = \begin{bmatrix} -.5 \\ -.5 \\ .9 \\ 1.1 \\ 3.3 \end{bmatrix}.$$

Data Fitting

We have

$$A^TA = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -3 & -1 & 1 & 2 & 3 \\ 9 & 1 & 1 & 4 & 9 \end{bmatrix} \begin{bmatrix} 1 & -3 & 9 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix} = \begin{bmatrix} 5 & 2 & 24 \\ 2 & 24 & 8 \\ 24 & 8 & 180 \end{bmatrix}$$

$(A^TA)^{-1} = \frac{1}{1876} \begin{bmatrix} 1064 & -42 & -140 \\ -42 & 81 & 2 \\ -140 & 2 & 29 \end{bmatrix}$, and the best solution in the sense of least square is

$$x^+ = \begin{bmatrix} t^+ \\ s^+ \\ r^+ \end{bmatrix} = (A^TA)^{-1}A^Tb.$$

Data Fitting

$$= (A^T A)^{-1} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -3 & -1 & 1 & 2 & 3 \\ 9 & 1 & 1 & 4 & 9 \end{bmatrix} \begin{bmatrix} -.5 \\ -.5 \\ .9 \\ 1.1 \\ 3.3 \end{bmatrix} = \begin{bmatrix} .246 \\ .569 \\ .109 \end{bmatrix}.$$

So $t^+ = .246, s^+ = .569, r^+ = .109$. The "best" parabola has the equation

$$y = .109x^2 + .569x + .246.$$

This is the parabola for which the sum of the squares of the deviations of the points from the parabola is a minimum.

Exercises 37.

1. *Find the equation of the line that best fits the sets of points* $(0,0),(1,1),(3,12)$ *in the sense of least squares.*
2. *Find the equation of the parabola that best fits the sets of points* $(0,0),(1,1),(-1,2),(2,2)$ *in the sense of least squares.*
3. *Find the cubic polynomial* $f(x)=px^3+qx^2+rx+s$ *that best fits the points* $(-2,0),(-1,-\frac{1}{2}),(0,1),(1,\frac{1}{2}),(2,-1)$ *in the sense of least squares.*
4. *Find a function of the form* $f(x)=ax+b2^x$ *that best fits the points* $(-1,1),(0,0),(2,-1)$.
5. *Find the center and radius of the best circle, in the sense of least squares, which fits the sets of points* $(0,0),(1,1),(3,-2)$.
6. *Find a curve with equation of the form* $y=a\sin x+b\sin 2x$ *that best fits the points* $(-\frac{\pi}{3},-4),(\frac{\pi}{6},2),(\frac{5\pi}{6},-1)$.

Solution for Exercises 37 are given below.

1. The "best" line has equation $y = \frac{59}{14}x - \frac{9}{7}$.
2. The best parabola has equation $y = \frac{3}{4}x^2 - \frac{13}{20}x + \frac{9}{20}$.
3. The best cubic is $f(x) = -\frac{1}{4}x^3 - \frac{2}{7}x^2 + \frac{3}{4}x + \frac{4}{7}$.
4. The best function is $f(x) = -\frac{17}{20}x + \frac{1}{6}2^x \approx 0.85x + 0.17(2^x)$.
5. The center of the best circle is $\left(\frac{17}{10}, -\frac{7}{10}\right)$ and the radius is $\frac{13}{5\sqrt{2}}$.
6. The best curve has equation
   $y = \frac{7\sqrt{3}+9}{6}\sin x + \frac{13\sqrt{3}-9}{18}\sin 2x \approx 3.52\sin x + 0.75\sin 2x$.

Exercises 38.

1. *Explain how you might find the "best" plane with equation of the form* $ax + by + cz = 1$ *through a given set of points,* $(x_1, y_1, z_1), (x_2, y_2, z_2), \ldots, (x_n, y_n, z_n)$.
2. *Describe a "sum of squares of deviations" test that could be used to estimate how good your plane is.*
3. *Find the "best" plane with equation of the form* $ax + by + cz = 1$ *through the points* $(2, 4, 6), (-1, 0, 12), (3, 7, 12), (4, 1, -6)$. *Give the coefficients to two decimal places of accuracy and, in each case, find the sum of squares of the deviations. (This question assumes that some computing power is available.)*

Solution : The best plane has equation $0.51x - 0.27y + 0.13z = 1$. The sum of squares of deviations is 7.35.

Application of Linear Algebra in Conic Sections

Definition 39.

A quadratic form in n variables $x_1, x_2, \ldots, x_n$ is a linear combination of the products $x_i x_j$, that is, a linear combination of squares $x_1^2, x_2^2, \ldots, x_n^2$ and cross terms $x_1x_2, x_1x_3, \ldots, x_1x_n, x_2x_3, \ldots, x_2x_n, \ldots, x_{n-1}x_n$.

Example 40.

- $q = x^2 - y^2 + 4xy$ *and* $q = x^2 + 3y^2 - 2xy$ *are quadraic forms in* x *and* y*;*
- $q = -4x_1^2 + x_2^2 + 4x_3^2 + 6x_1x_3$ *is a quadratic form in* x_1, x_2 *and* x_3*;*
- *the most general quadratic form in* x_1, x_2, x_3 *is*

$$a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + a_{12}x_1x_2 + a_{13}x_1x_3 + a_{23}x_2x_3.$$

Quadratic Forms

In the study of quadratic forms, our starting point is the observation that any quadratic form can be written in the form $x^T Ax$ where A is a symmetric matrix and $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$.

The matrix A is obtained by putting the coefficient of x_i^2 in the i-th diagonal position and splitting equally the coefficient of $x_i x_j$ between the (i, j) and (j, i) positions; that is, putting one half the coefficient in each position.

Example 41.

Suppose $q = x_1^2 - x_2^2 + 4x_1x_2$. *The coefficients of* x_1^2 *and* x_2^2 *are* 1 *and* -1*, respectively, so we put these, in order, into the two diagonal positions of a matrix* A. *The coefficient of* x_1x_2 *is* 4*, which we split equally between the* $(1,2)$ *and* $(2,1)$ *positions, putting a* 2 *in each place.*

So $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$ and, with $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, then $x^T A x = x_1^2 - x_2^2 + 4x_1x_2 = q$.
Also,

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1^2 - x_2^2 + 4x_1x_2.$$

Quadratic Forms

Here are some other examples.

- $4x^2 - 7xy + y^2 = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 4 & -\frac{7}{2} \\ -\frac{7}{2} & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$
- $4x_1^2 + x_2^2 + 4x_3^2 + 6x_1x_3 - 10x_2x_3 = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} -4 & 0 & 3 \\ 0 & 1 & -5 \\ 3 & -5 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$
- $2x_1^2 - 3x_2^2 + 7x_3^2 = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

When a quadratic form $q = \lambda_1 x_1^2 + \lambda_2 x_2^2 + \cdots + \lambda_n x_n^2$ has no cross terms, the matrix A giving $q = x^T Ax$ is a diagonal matrix, with diagonal entries the coefficients of $x_1^2, x_2^2, \ldots, x_n^2$.

$$\lambda_1 x_1^2 + \lambda_2 x_2^2 + \cdots + \lambda_n x_n^2 = \begin{bmatrix} x_1 & x_2 & \ldots & x_n \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

In general, the matrix A in $x^T Ax$ is not diagonal, but it is symmetric, so it can be orthogonality diagonalized (this is the Principal Axes Theorem).

Theorem 42 (Principal Axes Theorem).

A real symmetric matrix is orthogonally diagonalizable.

So, there exists an orthogonal matrix Q such that $Q^T AQ = D$ is a real diagonal matrix.

Set $y = Q^T x (= Q^{-1}x)$. Then $x = Qy$ and $x^T Ax = y^T Q^T AQy = y^T Dy$.

If the diagonal entries of D are $\lambda_1, \lambda_2, \ldots, \lambda_n$, the form $q = x^T Ax$ becomes

$$q = y^T Dy = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2.$$

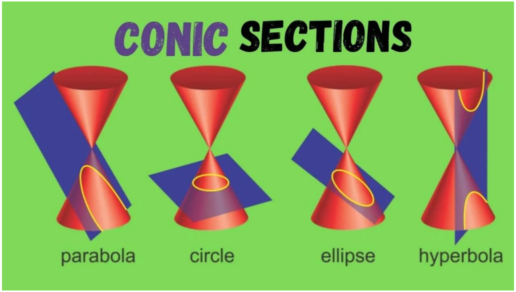
Example 43.

- *The quadratic form* $q = x^2 - y^2 + 4xy = x^T Ax$ *with* $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$.

  The eigenvalues of A *are* $\lambda_1 = \sqrt{5}$, $\lambda_2 = -\sqrt{5}$, *so* $q = x^T Ax = y^T Dy = \sqrt{5}y_1^2 - \sqrt{5}y_2^2$ *with* $y = Q^T x$ *for a certain matrix* Q.
- *The quadratic form* $q = -4x_1^2 + x_2^2 + 4x_3^2 + 6x_1x_3 = x^T Ax$ *with* $A = \begin{bmatrix} -4 & 0 & 3 \\ 0 & 1 & 0 \\ 3 & 0 & 4 \end{bmatrix}$. *The eigenvalues of* A *are* $1, 5, -5$, *so* $q = y_1^2 + 5y_2^2 - 5y_3^2$, *with* $y = Q^T x$ *for a certain matrix* Q.

Conic Sections

The conic sections are the nondegenerate curves generated by the intersections of a plane with one or two nappes of a cone. For a plane perpendicular to the axis of the cone, a circle is produced. For a plane that is not perpendicular to the axis and that intersects only a single nappe, the curve produced is either an ellipse or a parabola. The curve produced by a plane intersecting both nappes is a hyperbola. The ellipse and hyperbola are known as central conics.



Conic Sections

Remember that the *coordinates* of a vector x relative to a basis $\{v_1, v_2, \ldots, v_n\}$ are the scalars $c_1, c_2, \ldots, c_n$ required as coefficients when x is written as a linear combination of $v_1, v_2, \ldots, v_n$.

The coordinates of $\begin{bmatrix} -3 \\ 4 \end{bmatrix}$ relative to the standard basis are -3 and 4 because $\begin{bmatrix} -3 \\ 4 \end{bmatrix} = -3 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 4 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = -3e_1 + 4e_2$. Let us examine the equation $y = Q^T x$ in more detail with $y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ and $Q = \begin{bmatrix} q_1 & q_2 \\ \downarrow & \downarrow \\ & \end{bmatrix}$.

Since $Q^T = Q^{-1}$, $x = Qy = y_1 q_1 + y_2 q_2$, so x has coordinates y_1 and y_2 relative to the basis $\{q_1, q_2\}$.

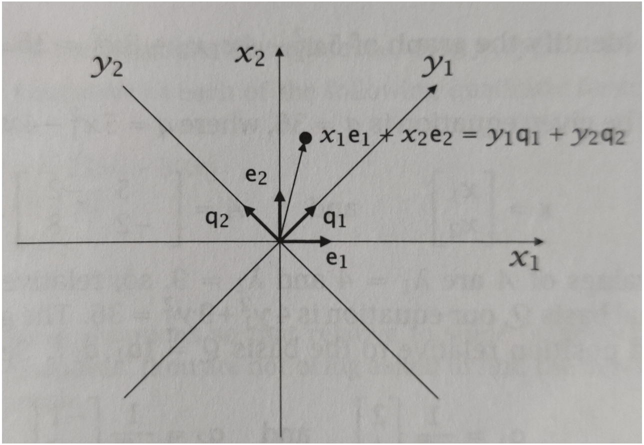
The set $Q = \{q_1, q_2\}$ is an orthonormal basis for $\mathbb{R}^2$ just like the standard basis $\{e_1, e_2\}$. A graph in $\mathbb{R}^2$ has an equation relative to the standard basis and another relative to Q – same graph, different equations.

For example, suppose $q_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$ and $q_2 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$. The relationship between the coordinates x_1, x_2 of a vector relative to the standard basis and its coordinates y_1, y_2 relative to Q is expressed by $x = Qy$:

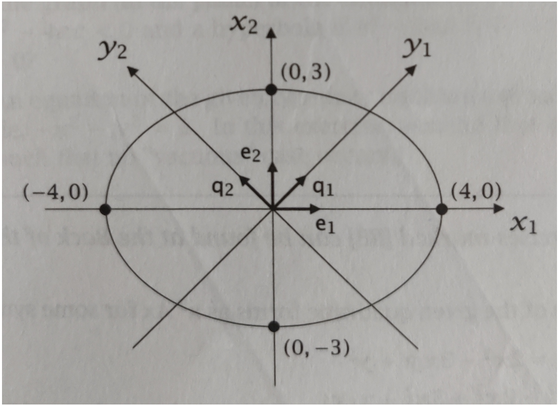
$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix},$$

so

$$x_1 = \frac{1}{\sqrt{2}} y_1 - \frac{1}{\sqrt{2}} y_2 \quad \text{and} \quad x_2 = \frac{1}{\sqrt{2}} y_1 + \frac{1}{\sqrt{2}} y_2.$$



The figure shows that the coordinates of (x_1, x_2) relative to the orthonormal basis $\{q_1, q_2\}$ are (y_1, y_2).



The ellipse with equation $\frac{x_1^2}{16} + \frac{x_2^2}{9} = 1$ relative to the standard basis is shown in the figure.

Relative to the basis Q, it has equation

$$\frac{1}{16}\left(\frac{1}{\sqrt{2}}y_1 - \frac{1}{\sqrt{2}}y_2\right)^2 + \frac{1}{9}\left(\frac{1}{\sqrt{2}}y_1 + \frac{1}{\sqrt{2}}y_2\right)^2 = 1,$$

that is, $25y_1^2 + 25y_2^2 + 14y_1y_2 = 288$.

This example is only for the purposes of illustration; if we had to choose between the equations $\frac{x_1^2}{16} + \frac{x_2^2}{9} = 1$ and $25y_1^2 + 25y_2^2 + 14y_1y_2 = 288$, most of us would choose the former, which has no cross terms.

Conic Sections

On the other hand, if a quadratic form happens to come to us with some nonzero cross terms, it would be nice to find a new set of "principal axes" relative to which the cross terms disappear.

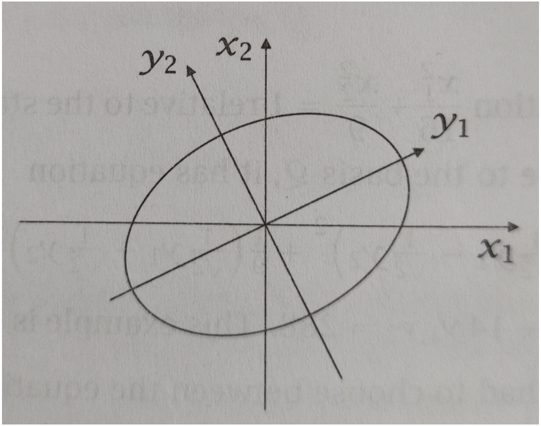
Let us identify the graph of $5x_1^2 - 4x_1x_2 + 8x_2^2 = 36$.

The given equation is $q = 36$, where $q = 5x_1^2 - 4x_1x_2 + 8x_2^2 = x^TAx$ with

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 5 & -2 \\ -2 & 8 \end{bmatrix}.$$

The eigenvalues of A are $\lambda_1 = 4$ and $\lambda_2 = 9$, so, relative to a certain new orthonormal basis Q, our equation is $4y_1^2 + 9y_2^2 = 36$.

The graph is an ellipse, in standard position relative to the basis $Q = \{q_1, q_2\}$. See the following figure.



The vectors

$$q_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \text{and} \quad q_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

are eigenvectors of A corresponding to $\lambda_1 = 4$ and $\lambda_2 = 9$, respectively.

Definition 44.

Let $A = (a_{ij})$ *be an* $n \times n$ *real matrix. The function* $q : \mathbb{R}^n \to \mathbb{R}$ *defined by:*

$$q_A(x) := Q_A((x_1, x_2, \ldots, x_n)^t) := \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j$$

is called the quadratic form *associated with A.*

If $A = diag(\lambda_1, \lambda_2, \ldots, \lambda_n)$, then $q(x) = \lambda_1 x_1^2 + \lambda_2 x_2^2 + \cdots + \lambda_n x_n^2$ is called a **diagonal form**.

Proposition 45.

$$q(x) = [x_1, x_2, \ldots, x_n]A \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x^T Ax \quad \textit{where} \quad x = (x_1, x_2, \ldots, x_n)^T.$$

Proof:

$$[x_1, x_2, \ldots, x_n]A \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = [x_1, x_2, \ldots, x_n] \begin{bmatrix} \sum_{j=1}^{n} a_{1j}x_j \\ \vdots \\ \sum_{j=1}^{n} a_{nj}x_j \end{bmatrix}$$

$$= \sum_{j=1}^{n} a_{1j}x_j x_1 + \cdots + \sum_{j=1}^{n} a_{nj}x_j x_n = \sum_{j=1}^{n}\sum_{i=1}^{n} a_{ij}x_i x_j = q(x).$$

Notice that A and B give rise to same $q(x)$ and $B = \frac{1}{2}(A + A^T)$ is a symmetric matrix.

Example 46.

- $A = \begin{bmatrix} 1 & 1 \\ 3 & 5 \end{bmatrix}, x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. *Then*

$$x^T A x = [x_1, x_2] \begin{bmatrix} 1 & 1 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = [x_1, x_2] \begin{bmatrix} x_1 + x_2 \\ 3x_1 + 5x_2 \end{bmatrix} = x_1^2 + 4x_1x_2 + 5x_2^2.$$

- *Let* $B = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}, x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. *Then*

$$x^T B x = [x_1, x_2] \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = [x_1, x_2] \begin{bmatrix} x_1 + 2x_2 \\ 2x_1 + 5x_2 \end{bmatrix} = x_1^2 + 4x_1x_2 + 5x_2^2.$$

Proposition 47.

For any $n \times n$ matrix A and the column vector $x = (x_1, x_2, \ldots, x_n)^T$

$$x^T Ax = x^T Bx \quad \textit{where} \quad B = \frac{1}{2}(A + A^T).$$

Hence every quadratic form is associated with a symmetric matrix.

Proof. $x^T Ax$ is a 1×1 matrix. Hence $x^T Ax = (x^T Ax)^T = x^T A^T x$. Hence

$$x^T Ax = \frac{1}{2}x^T Ax + \frac{1}{2}x^T A^T x = x^T \frac{1}{2}(A + A^T)x = x^T Bx.$$

We now show how the spectral theorem helps us in converting a quadratic form into a diagonal form.

Theorem 48.

Let $x^T Ax$ be a quadratic form associated with a real symmetric matrix A. Let U an orthogonal matrix such that $U^T AU = diag(\lambda_1, \lambda_2, \ldots, \lambda_n)$. Then

$$x^T Ax = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2.$$

where $y = (y_1, \ldots, y_n)$ is defined by

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = U \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = Uy.$$

Proof: Since $x = Uy$,

$$x^T Ax = (Uy)^T A(Uy) = y^T(U^T AU)y.$$

Since $U^T AU = diag(\lambda_1, \lambda_2, \ldots, \lambda_n)$, we get

$$x^T Ax = [y_1, y_2, \ldots, y_n] \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

$$= \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2.$$

Quadratic forms: An example

Let us determine the orthogonal matrix U which reduces the quadratic form $q(x) = 2x_1^2 + 4xy + 5x_2^2$ to a diagonal form. We write

$$q(x) = [x, y] \begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = x^T Ax.$$

The symmetric matrix A can be diagonalized. The eigenvalues of A are $\lambda_1 = 1$ and $\lambda_2 = 6$. An orthonormal set of eigenvectors for λ_1 and λ_2 is

$$v_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ -1 \end{bmatrix} \quad \text{and} \quad v_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Hence $U = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$. The change of variables equations are $\begin{bmatrix} x \\ y \end{bmatrix} = U \begin{bmatrix} u \\ v \end{bmatrix}$.
The diagonal form is:

$$[u, v] \begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix} [u, v]^T = u^2 + 6v^2.$$

Check that $U^T AU = diag(1, 6)$.

A conic section is the locus in the Cartesian plane $\mathbb{R}^2$ of an equation of the form

$$ax^2 + bxy + cy^2 + dx + ey + f = 0. \tag{14}$$

It can be proved that this equation represents one of the following :

(i) the empty set
(ii) a single point
(iii) one or two straight lines
(iv) an ellipse
(v) a hyperbola, or
(vi) a parabola.

The second degree part of (14), $q(x, y) = ax^2 + bxy + cy^2$ is a quadratic form. This determines the type of the conic.

We can write the equation into matrix form:

$$[x, y] \begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + [d, e] \begin{bmatrix} x \\ y \end{bmatrix} + f = 0.$$

Write $A = \begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix}$. Let $U = [u, v]$ be an orthogonal matrix whose column vectors u and v are eigenvectors of A with eigenvalues λ_1 and λ_2. Apply the change of variables

$$x = \begin{bmatrix} x \\ y \end{bmatrix} = U \begin{bmatrix} u \\ v \end{bmatrix}$$

to diagonalize the quadratic form $q(x, y)$ to the diagonal form $\lambda_1 u^2 + \lambda_2 v^2$.

Conic Sections

The orthonormal basis $\{u, v\}$ determines a new set of coordinate axes with respect to which the locus of the equation

$$[x, y]A[x, y]^T + B[x, y]^T + f = 0$$

with $B = [d, e]$ is same as the locus of the equation

$$0 = [u, v]diag(\lambda_1, \lambda_2)[u, v]^T + (BU)[u, v]^T + f$$

hence

$$\lambda_1 u^2 + \lambda_2 v^2 + [d, e][u, v][u, v]^T + f = 0. \tag{15}$$

Conic Sections

If the conic determined by (15) is not degenerate i.e., not an empty set, a point, nor line(s), then the signs of λ_1 and λ_2 determine whether it is a parabola, an hyperbola or an ellipse. The equation (14) will represent

- ellipse if $\lambda_1\lambda_2 > 0$
- hyperbola if $\lambda_1\lambda_2 < 0$
- parabola if $\lambda_1\lambda_2 = 0$.

Exercise 49.

Identify the graph of $2x^2 + 4xy + 5y^2 + 4x + 13y - 1/4 = 0.$

We have earlier diagonalized the quadratic form $2x^2 + 4xy + 5y^2$. The associated symmetric matrix, the eigenvectors and eigenvalues are displayed in the equation of diagonalization:

$$U^t AU = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix} \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix}.$$

Put $t = 1/\sqrt{5}$ for convenience. Then the change of coordinates equations are:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2t & t \\ -t & 2t \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix},$$

i.e., $x = t(2u + v)$ and $y = t(-u + 2v)$.

Substitute these into the original equation to get

$$u^2 + 6v^2 - \sqrt{5}u + 6\sqrt{5}v - \frac{1}{4} = 0.$$

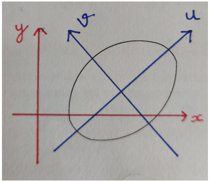
Complete the square to write this as

$$(u - \frac{1}{2}\sqrt{5})^2 + 6(v + \frac{1}{2}\sqrt{5})^2 = 9.$$

This is an equation of ellipse with center $(\frac{1}{2}\sqrt{5}, -\frac{1}{2}\sqrt{5})$ in the uv-plane.

Conic Sections: Examples

The u-axis and v-axis are determined by the eigenvectors v_1 and v_2 as indicated in the following figure:



Exercise 50.

Identify the graph of $2x^2 - 4xy - y^2 - 4x + 10y - 13 = 0$.

Here, the matrix $A = \begin{bmatrix} 2 & -2 \\ -2 & -1 \end{bmatrix}$ gives the quadratic part of the equation. We write the equation in matrix form as

$$[x, y] \begin{bmatrix} 2 & -2 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + [-4, 10] \begin{bmatrix} x \\ y \end{bmatrix} - 13 = 0.$$

Let $t = 1/\sqrt{5}$. The eigenvalues of A are $\lambda_1 = 3$, $\lambda_2 = -2$. An orthonormal set of eigenvectors is $u = t(2, -1)^t$ and $v = t(1, 2)^t$. Put

$$U = t \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} x \\ y \end{bmatrix} = U \begin{bmatrix} u \\ v \end{bmatrix}.$$

The transformed equation becomes

$$3u^2 - 2v^2 - 4t(2u + v) + 10t(-u + 2v) - 13 = 0$$

which is the same as $3u^2 - 2v^2 - 18tu + 16tv - 13 = 0$.

Complete the square in u and v to get

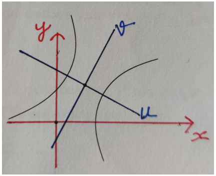
$$3(u - 3t)^2 - 2(v - 4t)^2 = 12$$

or

$$\frac{(u - 3t)^2}{4} - \frac{(v - 4t)^2}{6} = 1.$$

Conic Sections

This represents a hyperbola with center $(3t, 4t)$ in the uv-plane. The eigenvectors u and v determine the directions of positive u and v axes.



Exercise 51.

Identify the graph of $9x^2 + 24xy + 16xy^2 - 20x + 15y = 0.$

The symmetric matrix for the quadratic part is $A = \begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix}$. The eigenvalues are $\lambda_1 = 25$, $\lambda_2 = 0$. An orthonormal set of eigenvectors is $u = a(3,4)^t$, $v = a(-4,3)^t$, where $a = 1/5$. An orthogonal diagonalizing matrix is $U = a\begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}$. The equations of change of coordinates are

$$\begin{bmatrix} x \\ y \end{bmatrix} = U \begin{bmatrix} u \\ v \end{bmatrix} \quad \text{i.e.,} \; x = a(3u - 4v),\; y = a(4u + 3v).$$

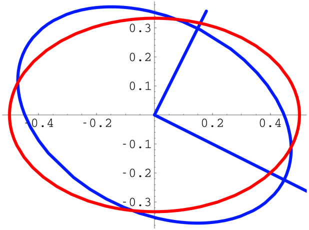
The equation in uv-plane is $u^2 + v = 0$. This is an equation of parabola with its vertex at the origin.

Drawing Conics

To plot the ellipse

$$\frac{X^2}{a^2} + \frac{Y^2}{b^2} = 1$$

centred at the origin, we construct a box of 'semi-width' a and 'semi-height' b, and fit the ellipse tightly into it. In the following figure, we take $a = \frac{1}{2}$ and $b = \frac{1}{3}$, and the result is shown below in red.



Drawing Conics

The clockwise rotation R_θ transforms a point $\begin{pmatrix} X \\ Y \end{pmatrix}$ on the red conic to a point $\begin{pmatrix} x \\ y \end{pmatrix}$ of the blue one. It follows that the blue conic represents the locus of points (x, y) satisfying the original equation

$$5x^2 + 4xy + 8y^2 = 1,$$

with the black axes now representing the old coordinates x, y.

Conic Sections: Examples

A hyperbola

$$\frac{X^2}{a^2} - \frac{Y^2}{b^2} = 1$$

can be drawn in a similar way, by first constructing the box with centre the origin and size $2a \times 2b$. The diagonal lines form the *asymptotes* of the hyperbola, and taken together they have equation

$$\frac{X^2}{a^2} - \frac{Y^2}{b^2} = 0, \quad \text{or} \quad Y = \pm\frac{b}{a}X.$$

The two branches of the hyperbola itself are now easily drawn, and pass through the points $(-a, 0)$ and $(0, a)$ on the horizontal axis.

Example 52.

Let c be a constant. The conic $4xy + 3y^2 = c$ is associated to the symmetric matrix

$$S = \begin{pmatrix} 0 & 2 \\ 2 & 3 \end{pmatrix}.$$

Since $\det(S) < 0$ the eigenvalues have opposite signs (they are 4 and -1) and the conic is a hyperbola provided $c \neq 0$. The matrix S is diagonalized by

$$P = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix},$$

and this corresponds to a rotation by $\theta = +25.6^\circ$.

Exercises 53.

1. *Write the given quadratic form* $q = 2x^2 - 3xy + y^2$ *as* $x^T Ax$ *for some symmetric matrix* A.
2. *Let* $q_1 = \frac{1}{5}\begin{bmatrix} 3 \\ -4 \end{bmatrix}$ *and* $q_2 = \frac{1}{5}\begin{bmatrix} 4 \\ 3 \end{bmatrix}$. *Let* (x_1, x_2) *be the coordinates of the point* P *in the Euclidean plane relative to the standard basis and* (y_1, y_2) *the coordinates of* P *relative to the orthogonal basis* $Q = \{q_1, q_2\}$.
   2.1 *Write down the equations that relate* x_1, x_2 *to* y_1, y_2.
   2.2 *Find the equations of the quadratic form* $q = 25x_1^2 - 50x_2^2$ *in terms of* y_1 *and* y_2.
3. *Rewrite the given equation* $6x^2 - 4xy + 3y^2 = 1$ *so that relative to some orthonormal basis the cross terms disappear. (You are not being asked to find the new basis.) Try to identify the graph.*
4. *Find an orthonormal basis of* $\mathbb{R}^3$ *relative to which the quadratic form* $q = 3x_1^2 + 4x_2^2 + x_3^2 - 4x_2x_3$ *has no cross terms. Express* q *relative to this new basis.*

Solution for Exercises 53 are given below.

1. $q = x^T A x$ with $x = \begin{bmatrix} x \\ y \end{bmatrix}$ and $A = \begin{bmatrix} 2 & -\frac{3}{2} \\ -\frac{3}{2} & 1 \end{bmatrix}$.
2. $q = -23y_1^2 + 72y_1y_2 - 2y_2^2$.
3. $2x^2 + 7y^2 = 1$. The graph is an ellipse.
4. $q_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, q_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} q_3 = \frac{1}{\sqrt{5}} \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$ and $q = 3y_2^2 + 5y_3^2$.

Application of Linear Algebra in Satellite Motion

Introduction

Control theory deals with the maneuver of the state or trajectory of the system, modelled by ODE. Some of the important properties of such systems are **controllability, observability** and **optimality**.

Notation :

$L_2([t_0, t_1], \mathbb{R}^m)$	the space of vectors $\vec{u} \in \mathbb{R}^m$ in which each component is a function in $L_2([t_0, t_1])$

Controllability

Let us consider the linear system of the form

$$\frac{d\vec{x}}{dt} = A(t)\vec{x}(t) + B(t)\vec{u}(t), \quad \vec{x}(t) = \vec{x}_0 \tag{16}$$

where $A(t)$ is an $n \times n$ matrix, $B(t)$ is an $n \times m$ matrix, $\vec{u}(t) \in \mathbb{R}^m$ and $\vec{x}(t) \in \mathbb{R}^n$.

$\vec{u}(t)$ is called **control** or **input vector** and $\vec{x}(t)$ the corresponding **trajectory or state of the system.**

The typical controllability problem involves the determination of the control vector $\vec{u}(t)$ such that the state vector $\vec{x}(t)$ has the desired properties.

We assume that the entries of the matrices $A(t)$, $B(t)$ are continuous so that the above system has a unique solution $\vec{x}(t)$ for a given input $\vec{u}(t)$.

Definition 54.

The linear system given by Equation (16) is said to be controllable *if given any initial state $\vec{x}_0$ and any final state $\vec{x}_f$ in $\mathbb{R}^n$, there exist a control $\vec{u}(t)$ so that the corresponding trajectory $\vec{x}(t)$ of Equation (16) satisfies the condition*

$$\vec{x}(t_0) = \vec{x}_0, \qquad \vec{x}(t_f) = \vec{x}_f.$$

The control $\vec{u}(t)$ is said to steer the trajectory from the initial state $\vec{x}_0$ to the final state $\vec{x}_f$.

Controllability

If an input to a system can be found that takes every state variable from a desired initial state to a desired final state, the system is said to be controllable; otherwise, the system is uncontrollable.

The ability to control all of the state variables is a requirement for the design of a controller. State-variable feedback gains cannot be designed if any state variable is uncontrollable.

Uncontrollability can be viewed best with diagonalized systems.

Controllability

Assume that $\phi(t, t_0)$ is the **transition matrix** of the above system. Then by the variation of parameter formula, Equation (16) is equivalent to the following integral equation

$$\vec{x}(t) = \phi(t, t_0)\vec{x}_0 + \int_{t_0}^{t} \phi(t, \tau)B(\tau)\vec{u}(\tau)d\tau.$$

As $(t, \tau) \to \phi(t, \tau)$ is continuous, it follows that

$$\|\phi(t, \tau)\| \leq M$$

for all $t, \tau \in [t_0, t_f]$.

So controllability of Equation (16) is equivalent to finding $\vec{u}(t)$ such that

$$\vec{x}_f = \vec{x}(t_f) = \phi(t_f, t_0)\vec{x}_0 + \int_{t_0}^{t_f} \phi(t_f, \tau)B(\tau)\vec{u}(\tau)d\tau.$$

Equivalently,

$$\vec{x}(t_f) - \phi(t_f, t_0)\vec{x}_0 = \int_{t_0}^{t_f} \phi(t_f, \tau)B(\tau)\vec{u}(\tau)d\tau. \tag{17}$$

Controllability

Let us define a linear operator $L : X = L_2([t_0, t_f], \mathbb{R}^m)$ by

$$[L\vec{u}] = \int_{t_0}^{t_f} \phi(t_f, \tau)B(\tau)\vec{u}(\tau)d\tau.$$

Then, in view of (17), the controllability problem reduces to showing that operator L is surjective.

Theorem 55.

The system given by Equation (16) is controllable iff the controllability Grammian

$$W(t_0, t_f) = \int_{t_0}^{t_f} [\phi(t_f, \tau)B(\tau)B^\tau(\tau)\phi^\tau(t, t_f)]d\tau$$

is nonsingular. A control $\vec{u}(t)$ steering the system from the initial state $\vec{x}_0$ to the final state $\vec{x}_f$ is given by

$$\vec{u}(t) = B^\tau(t)\phi^\tau(t_f, t)[W(t_0, t_f)]^{-1}[\vec{x}_f - \phi(t_f, t_0)\vec{x}_0].$$

Controllability

The controllability Grammian $W(t_0, t_f)$ has some interesting properties, which we now describe.

Theorem 56.

(i) *$W(t_0, t_f)$ is symmetric and positive semidefinite.*

(ii) *$W(t_0, t_f)$ satisfies the linear differential equation*

$$\frac{d}{dt}[W(t_0,t)] = A(t)W(t_0,t) + W(t_0,t)A^{\tau}(t) + B(t)B^{\tau}(t)$$
$$W(t_0,t_0) = 0$$

(iii) *$W(t_0, t_f)$ satisfies functional equation*

$$W(t_0,t_f) = W(t_0,t) + \phi(t_f,t)W(t,t_f)\phi^{\tau}(t_f,t).$$

Remark 57.

Theorem 55 implies that to check the controllability of the linear system given by Equation (16), one needs to verify the invertibility of the Grammian matrix $W(t_0, t_f)$*. This is a very tedious task. However, if* $A(t)$ *and* $B(t)$ *are time invariant matrices* A, B*, then the controllability of the linear system is obtained in terms of the rank of the following* controllability matrix

$$C = [B, AB, \dots, A^{n-1}B] \tag{18}$$

The next theorem, in this direction, is due to Kalman.

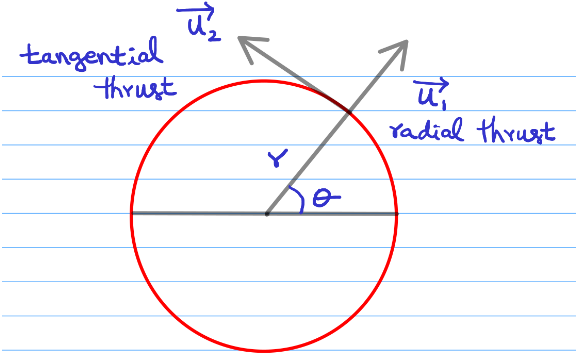
Theorem 58.

The linear autonomous system

$$\frac{d\vec{x}}{dt} = A\vec{x}(t) + B\vec{u}(t), \quad \vec{x}(t_0) = \vec{x}_0 \tag{19}$$

is controllable iff the controllability matrix C given by Equation (18) has rank n.

Linearized Motion of Satellite Orbiting Around the Earth



Example 59.

Let us consider the linearized motion of a satellite of unit mass orbiting around the earth. This is given by the control system of the form given by Equation (19), where

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3\omega^2 & 0 & 0 & 2\omega \\ 0 & 0 & 0 & 1 \\ 0 & -2\omega & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

For this system, one can easily compute the controllability matrix, $C = [B, AB, A^2B, A^3B]$. It is given by

$$C = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 2\omega & -\omega^2 & 0 \\ 1 & 0 & 0 & 2\omega & -\omega^2 & 0 & 0 & -2\omega^3 \\ 0 & 0 & 0 & 1 & -2\omega & 0 & 0 & -4\omega^2 \\ 0 & 1 & -2\omega & 0 & 0 & -4\omega^2 & 2\omega^3 & 0 \end{bmatrix}.$$

One can verify that rank of C is 4 and hence the linearized motion of the satellite is controllable. It is interesting to ask the following question. What happens when one of the controls or thrusts becomes inoperative ?

Solution of the Control System of Satellite Motion Problem

For this purpose set $u_2 = 0$ and hence B reduces to $B_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}^T$. So, the controllability matrix $C_1 = [B_1, AB_1, A^2B_1, A^3B_1]$, is given by

$$C_1 = \begin{bmatrix} 0 & 1 & 0 & -\omega^2 \\ 1 & 0 & -\omega^2 & 0 \\ 0 & 0 & -2\omega & 0 \\ 0 & -2\omega & 0 & 2\omega^3 \end{bmatrix}.$$

C_1 has rank 3.

Solution of the Control System of Satellite Motion Problem

On the other hand $u_1 = 0$ reduces B to $B_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^T$ and this gives controllability matrix $C_2 = [B_2, AB_2, A^2B_2, A^3B_2]$, as

$$C_2 = \begin{bmatrix} 0 & 0 & 2\omega & 0 \\ 0 & 2\omega & 0 & -2\omega^3 \\ 0 & 1 & 0 & -4\omega^2 \\ 1 & 0 & -4\omega^2 & 0 \end{bmatrix}.$$

C_2 has rank 4.

Conclusion

Since u_1 was radial thrust and u_2 was tangential thrust, we see that the loss of radial thrust does not destroy controllability whereas loss of tangential thrust does.

In terms of practical importance of satellite in motion, the above analysis means that we can maneuver the satellite just with radial rocket thrust.

Observability

Consider the input-output system of the form

$$\frac{d\vec{x}}{dt} = A(t)\vec{x}(t) + B(t)\vec{x}(t) \tag{20}$$

$$\vec{y}(t) = C(t)\vec{x}(t) \tag{21}$$

The questions concerning observability relate to the problem of determining the values of the state vector $\vec{x}(t)$, knowing only the output vector $\vec{y}(t)$ over some interval $I = [t_0, t_f]$ of time.

In other words, if the initial-state vector, $\vec{x}(t_0)$, can be found from $\vec{u}(t)$ and $y(t)$ measured over a finite interval of time from t_0, the system is said to be **observable**; otherwise the system is said to be **unobservable**.

Observability

$A(t) \in \mathbb{R}^{n \times n}$, $B(t) \in \mathbb{R}^{n \times m}$ are assumed to be continuous functions of t. Let $\phi(t, t_0)$ be the transition matrix of the above system.

Then the output vector $\vec{y}(t)$ can be expressed as

$$\vec{y}(t) = C(t)\phi(t, t_0)\vec{x}(t_0) + \vec{y}_1(t) \tag{22}$$

where $\vec{y}_1(t)$ is known quantity of the form

$$y_1(t) = \int_{t_0}^{t_f} C(t)\phi(t, \tau)B(\tau)\mu(\tau)d\tau.$$

Thus, from Equation (22) it follows that if we are concerned about the determination of $\vec{x}(t_0)$, based on the output $\vec{y}(t)$, we need only the homogeneous system

$$\frac{d\vec{x}}{dt} = A\vec{x}(t), \qquad \vec{y}(t) = C(t)\vec{x}(t) \tag{23}$$

in place of Equation (20).

Definition 60.

We shall say that Equation (23) is observable *on* $I = [t_0, t_f]$ *if* $\vec{y}(t) = 0$ *on* I *implies that* $\vec{x}(t) = 0$ *on* I.
We define a linear operator $L : \mathbb{R}^n \mapsto X = L_2([t_0, t_f], \mathbb{R}^n)$ *as*

$$[L\vec{x}_0](t) = C(t)\phi(t, t_0)\vec{x}_0 = H(t)\vec{x}_0$$

where $H : t \to H(t)$ *is a matrix function which is continuous in* t.
Observability of the system Equation (23), then, reduces to proving that L *is one-one.*

The following theorem gives the observability of the system given by Equation (23), in terms of the nonsingularity of the matrix $M(t_0, t_f)$.

Theorem 61.

For the homogeneous system given by Equation (23), it is possible to determine the initial state $\vec{x}(t_0)$ within an additive constant vector which lies with null space of $M(t_0, t_f)$ which is defined by

$$M(t_0, t_f) = \int_{t_0}^{t_f} \phi^T(t, t_0) C^T(t) C(t) \phi(t, t_0) dt. \tag{24}$$

Hence $\vec{x}(t_0)$ is uniquely determined if $M(t_0, t_f)$ is nonsingular. That is, Equation (23) is observable iff $M(t_0, t_f)$ is nonsingular.

$M(t_0, t_f)$ defined by Equation (24) is called the **observability Grammian**.

Theorem 62.

The observability Grammian $M(t_0, t_f)$ satisfies the following properties

(i) $M(t_0, t_f)$ *is symmetric and positive semidefinite.*

(ii) $M(t_0, t_f)$ *satisfies the following matrix differential equation*

$$\frac{d}{dt}[M(t, t_1)] = -A^T(t)M(t, t_1) - M(t, t_1)A(t) - C^T(t)C(t)$$
$$M(t_1, t_1) = 0.$$

(iii) $M(t_0, t_f)$ *satisfies the functional equation*

$$M(t_0, t_f) = M(t_0, t) + \phi^T(t, t_0)M(t, t_1)\phi(t, t_0).$$

We also have the following theorem giving the necessary and sufficient contition which is easily verifiable for observability.

Let us denote by O the **observability matrix**

$$O = [C, CA, \ldots, CA^{n-1}]. \tag{25}$$

Theorem 63.

The system given by Equation (23) is observable iff the observable matrix O given by Equation (25) is of rank n.

Example 64.

As a continuation of Example 59, consider the satellite orbiting around the Earth.

We assume that we can measure only the distance r of the satellite from the centre of the force and the angle θ. So, the defining state-output equation of the satellite is given by

$$\frac{d\vec{x}}{dt} = A\vec{x}(t), \qquad \vec{y}(t) = Cx(t)$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3\omega^2 & 0 & 0 & 2\omega \\ 0 & 0 & 0 & 1 \\ 0 & -2\omega & 0 & 0 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Observability

$\vec{x} = (x_1, x_2, x_3, x_4)$, $\vec{y} = (y_1, y_2)$; $y_1 = r$, $y_2 = \theta$ being the radial and angle measurements.

So, the observability matrix O is given by

$$O = [C, CA, CA^2, CA^3]$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 3\omega^2 & 0 & 0 & 2\omega \\ 0 & -2\omega & 0 & 0 \\ 0 & -\omega^2 & 0 & 0 \\ -6\omega^3 & 0 & 0 & -4\omega^2 \end{bmatrix}$$

Observability

Rank of C is 4 and hence the above state-output system is observable. To minimize the measurement, we might be tempted to measure y_1 only not y_2. This gives

$$C_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix}^T$$

and

$$O_1 = [C_1, C_1A, C_1A^2, C_1A^3] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 3\omega^2 & 0 & 0 & 2\omega \\ 0 & -\omega^2 & 0 & 0 \end{bmatrix}$$

which is of rank 3.

Observability

However, if y_1 is not measured, we get

$$C_2 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \end{bmatrix}^T$$

$$O_2 = [C_2, C_2A, C_2A^2, C_2A^3]$$

$$= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -2\omega & 0 & 0 \\ -6\omega^3 & 0 & 0 & -4\omega^2 \end{bmatrix}$$

which is of rank 4. Thus, the state of the satellite is known from angle measurements alone but this is not so for radial measurements.

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