Control Theory of Linear Systems

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Introduction

Control theory deals with the maneuver of the state or trajectory of the system, modelled by ODE. Some of the important properties of such systems are **controllability, observability** and **optimality**.

In this lecture we show how operator theoretic approach in function spaces is useful to discuss the above mentioned properties. We substantiate our theory by concrete examples.

Notations :

$\mathbb{R}^n$	n-dimensional Euclidean space
$L_2([t_0, t_1], \mathbb{R}^m)$	the space of vectors $\vec{u} \in \mathbb{R}^m$ in which each component is a function in $L_2([t_0, t_1])$

Controllability

Let us consider the linear system of the form

$$\frac{d\vec{x}}{dt} = A(t)\vec{x}(t) + B(t)\vec{u}(t), \quad \vec{x}(t) = \vec{x}_0 \tag{1}$$

where $A(t)$ is an $n \times n$ matrix, $B(t)$ is an $n \times m$ matrix, $\vec{u}(t) \in \mathbb{R}^m$ and $\vec{x}(t) \in \mathbb{R}^n$.

$\vec{u}(t)$ is called **control** or **input vector** and $\vec{x}(t)$ the corresponding **trajectory or state of the system.**

The typical controllability problem involves the determination of the control vector $\vec{u}(t)$ such that the state vector $\vec{x}(t)$ has the desired properties.

Controllability

We assume that the entries of the matrices $A(t)$, $B(t)$ are continuous so that the above system has a unique solution $\vec{x}(t)$ for a given input $\vec{u}(t)$.

Definition 1.

The linear system given by Equation (1) is said to be **controllable** *if given any initial state* $\vec{x}_0$ *and any final state* $\vec{x}_f$ *in* $\mathbb{R}^n$*, there exist a control* $\vec{u}(t)$ *so that the corresponding trajectory* $\vec{x}(t)$ *of Equation (1) satisfies the condition*

$$\vec{x}(t_0) = \vec{x}_0, \qquad \vec{x}(t_f) = \vec{x}_f.$$

The control $\vec{u}(t)$ *is said to* **steer the trajectory from the initial state $\vec{x}_0$ to the final state $\vec{x}_f$**.

Controllability

If an input to a system can be found that takes every state variable from a desired initial state to a desired final state, the system is said to be controllable; otherwise, the system is uncontrollable.

The ability to control all of the state variables is a requirement for the design of a controller. State-variable feedback gains cannot be designed if any state variable is uncontrollable.

Uncontrollability can be viewed best with diagonalized systems.

Controllability

Assume that $\phi(t, t_0)$ is the transition matrix of the above system. Then by the variation of parameter formula, Equation (1) is equivalent to the following integral equation

$$\vec{x}(t) = \phi(t, t_0)\vec{x}_0 + \int_{t_0}^{t} \phi(t, \tau)B(\tau)\vec{u}(\tau)d\tau.$$

As $(t, \tau) \to \phi(t, \tau)$ is continuous, it follows that

$$\|\phi(t, \tau)\| \leq M$$

for all $t, \tau \in [t_0, t_f]$.

Controllability

So controllability of Equation (1) is equivalent to finding $\vec{u}(t)$ such that

$$\vec{x}_f = \vec{x}(t_f) = \phi(t_f, t_0)\vec{x}_0 + \int_{t_0}^{t_f} \phi(t_f, \tau)B(\tau)\vec{u}(\tau)d\tau.$$

Equivalently,

$$\vec{x}(t_f) - \phi(t_f, t_0)\vec{x}_0 = \int_{t_0}^{t_f} \phi(t_f, \tau)B(\tau)\vec{u}(\tau)d\tau. \qquad (2)$$

Controllability

Let us define a linear operator $L : X = L_2([t_0, t_f], \mathbb{R}^m)$ by

$$[L\vec{u}] = \int_{t_0}^{t_f} \phi(t_f, \tau)B(\tau)\vec{u}(\tau)d\tau.$$

Then, in view of (2), the controllability problem reduces to showing that operator L is surjective.

Controllability

Theorem 2.

The system given by Equation (1) is controllable iff the **controllability Grammian**

$$W(t_0, t_f) = \int_{t_0}^{t_f} [\phi(t_f, \tau) B(\tau) B^{\tau}(\tau) \phi^{\tau}(t, t_f)] d\tau$$

is nonsingular. A control $\vec{u}(t)$ steering the system from the initial state $\vec{x}_0$ to the final state $\vec{x}_f$ is given by

$$\vec{u}(t) = B^{\tau}(t) \phi^{\tau}(t_f, t) [W(t_0, t_f)]^{-1} [\vec{x}_f - \phi(t_f, t_0) \vec{x}_0].$$

Controllability

The controllability Grammian $W(t_0, t_f)$ has some interesting properties, which we now describe.

Theorem 3.

(i) *$W(t_0, t_f)$ is symmetric and positive semidefinite.*

(ii) *$W(t_0, t_f)$ satisfies the linear differential equation*

$$\frac{d}{dt}[W(t_0, t)] = A(t)W(t_0, t) + W(t_0, t)A^{\tau}(t) + B(t)B^{\tau}(t)$$
$$W(t_0, t_0) = 0$$

(iii) *$W(t_0, t_f)$ satisfies functional equation*

$$W(t_0, t_f) = W(t_0, t) + \phi(t_f, t)W(t, t_f)\phi^{\tau}(t_f, t).$$

Controllability

Remark 4.

Theorem 2 implies that to check the controllability of the linear system given by Equation (1), one needs to verify the invertibility of the Grammian matrix $W(t_0, t_f)$. *This is a very tedious task. However, if* $A(t)$ *and* $B(t)$ *are time invariant matrices* A, B, *then the controllability of the linear system is obtained in terms of the rank of the following controllability matrix*

$$C = [B, AB, \dots, A^{n-1}B] \tag{3}$$

The next theorem, in this direction, is due to Kalman.

Theorem 5.

The linear autonomous system

$$\frac{d\vec{x}}{dt} = A\vec{x}(t) + B\vec{u}(t), \quad \vec{x}(t_0) = \vec{x}_0 \tag{4}$$

is controllable iff the controllability matrix C given by Equation (3) has rank n.

Example 6.

Let us consider the linearized motion of a satellite of unit mass orbiting around the earth. This is given by the control system of the form given by Equation (4), where

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3\omega^2 & 0 & 0 & 2\omega \\ 0 & 0 & 0 & 1 \\ 0 & -2\omega & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

For this system, one can easily compute the controllability matrix, $C = [B, AB, A^2B, A^3B]$. It is given by

$$C = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 2\omega & -\omega^2 & 0 \\ 1 & 0 & 0 & 2\omega & -\omega^2 & 0 & 0 & -2\omega^3 \\ 0 & 0 & 0 & 1 & -2\omega & 0 & 0 & -4\omega^2 \\ 0 & 1 & -2\omega & 0 & 0 & -4\omega^2 & 2\omega^3 & 0 \end{bmatrix}.$$

One can verify that rank of C is 4 and hence the linearized motion of the satellite is controllable. It is interesting to ask the following question.

What happens when one of the controls or thrusts becomes inoperative ?

Controllability

For this purpose set $u_2 = 0$ and hence B reduces to $B_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}^T$.
So, the controllability matrix $C_1 = [B_1, AB_1, A^2B_1, A^3B_1]$, is given by

$$C_1 = \begin{bmatrix} 0 & 1 & 0 & -\omega^2 \\ 1 & 0 & -\omega^2 & 0 \\ 0 & 0 & -2\omega & 0 \\ 0 & -2\omega & 0 & 2\omega^3 \end{bmatrix}.$$

C_1 has rank 3.

Controllability

On the other hand $u_1 = 0$ reduces B to $B_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^T$ and this gives controllability matrix $C_2 = [B_2, AB_2, A^2B_2, A^3B_2]$, as

$$C_2 = \begin{bmatrix} 0 & 0 & 2\omega & 0 \\ 0 & 2\omega & 0 & -2\omega^3 \\ 0 & 1 & 0 & -4\omega^2 \\ 1 & 0 & -4\omega^2 & 0 \end{bmatrix}.$$

C_2 has rank 4.

Since u_1 was radial thrust and u_2 was trangential thrust, we see that the loss of radial thrust does not destroy controllability where as loss of tangential thrust does. In terms of practical importance of satellite in motion, the above analysis means that we can maneuver the satellite just with radial rocket thrust.

Observability

Consider the input-output system of the form

$$\frac{d\vec{x}}{dt} = A(t)\vec{x}(t) + B(t)\vec{x}(t) \quad (5)$$

$$\vec{y}(t) = C(t)\vec{x}(t) \quad (6)$$

The questions concerning observability relate to the problem of determining the values of the state vector $\vec{x}(t)$, knowing only the output vector $\vec{y}(t)$ over some interval $I = [t_0, t_f]$ of time.

In other words, if the initial-state vector, $\vec{x}(t_0)$, can be found from $\vec{u}(t)$ and $y(t)$ measured over a finite interval of time from t_0, the system is said to be **observable**; otherwise the system is said to be **unobservable**.

Observability

$A(t) \in \mathbb{R}^{n\times n}$, $B(t) \in \mathbb{R}^{n\times m}$ are assumed to be continuous functions of t. Let $\phi(t, t_0)$ be the transition matrix of the above system.

Then the output vector $\vec{y}(t)$ can be expressed as

$$\vec{y}(t) = C(t)\phi(t, t_0)\vec{x}(t_0) + \vec{y}_1(t) \tag{7}$$

where $\vec{y}_1(t)$ is known quantity of the form

$$y_1(t) = \int_{t_0}^{t_f} C(t)\phi(t, \tau)B(\tau)\mu(\tau)d\tau.$$

Observability

Thus, from Equation (7) it follows that if we are concerned about the determination of $\vec{x}(t_0)$, based on the output $\vec{y}(t)$, we need only the homogeneous system

$$\frac{d\vec{x}}{dt} = A\vec{x}(t), \qquad \vec{y}(t) = C(t)\vec{x}(t) \tag{8}$$

in place of Equation (5).

Observability

Definition 7.

We shall say that Equation (8) is **observable** *on* $I = [t_0, t_f]$ *if* $\vec{y}(t) = 0$ *on* I *implies that* $\vec{x}(t) = 0$ *on* I.
We define a linear operator $L : \mathbb{R}^n \mapsto X = L_2([t_0, t_f], \mathbb{R}^n)$ *as*

$$[L\vec{x}_0](t) = C(t)\phi(t, t_0)\vec{x}_0 = H(t)\vec{x}_0$$

where $H : t \to H(t)$ *is a matrix function which is continuous in* t.
Observability of the system Equation (8), then, reduces to proving that L *is one-one.*

The following theorem gives the observability of the system given by Equation (8), in terms of the nonsingularity of the matrix $M(t_0, t_f)$.

Theorem 8.

For the homogeneous system given by Equation (8), it is possible to determine the initial state $\vec{x}(t_0)$ within an additive constant vector which lies with null space of $M(t_0, t_f)$ which is defined by

$$M(t_0, t_f) = \int_{t_0}^{t_f} \phi^T(t, t_0) C^T(t) C(t) \phi(t, t_0) dt. \tag{9}$$

Hence $\vec{x}(t_0)$ is uniquely determined if $M(t_0, t_f)$ is nonsingular. That is, Equation (8) is observable iff $M(t_0, t_f)$ is nonsingular.

$M(t_0, t_f)$ defined by Equation (9) is called the **observability Grammian**.

Observability

Theorem 9.

The observability Grammian $M(t_0, t_f)$ satisfies the following properties

(i) $M(t_0, t_f)$ *is symmetric and positive semidefinite.*

(ii) $M(t_0, t_f)$ *satisfies the following matrix differential equation*

$$\frac{d}{dt}[M(t, t_1)] = -A^T(t)M(t, t_1) - M(t, t_1)A(t) - C^T(t)C(t)$$
$$M(t_1, t_1) = 0.$$

(iii) $M(t_0, t_f)$ *satisfies the functional equation*

$$M(t_0, t_f) = M(t_0, t) + \phi^T(t, t_0)M(t, t_1)\phi(t, t_0).$$

Observability

We also have the following theorem giving the necessary and sufficient contition which is easily verifiable for observability.

Let us denote by O the **observability matrix**

$$O = [C, CA, \ldots, CA^{n-1}]. \tag{10}$$

Theorem 10.

The system given by Equation (8) is observable iff the observable matrix O given by Equation (10) is of rank n.

Example 11.

As a continuation of Example 6, consider the satellite orbiting around the Earth.

We assume that we can measure only the distance r of the satellite from the centre of the force and the angle θ. So, the defining state-output equation of the satellite is given by

$$\frac{d\vec{x}}{dt} = A\vec{x}(t), \qquad \vec{y}(t) = Cx(t)$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3\omega^2 & 0 & 0 & 2\omega \\ 0 & 0 & 0 & 1 \\ 0 & -2\omega & 0 & 0 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Observability

$\vec{x} = (x_1, x_2, x_3, x_4)$, $\vec{y} = (y_1, y_2)$; $y_1 = r$, $y_2 = \theta$ being the radial and angle measurements.

So, the observability matrix O is given by

$$O = [C, CA, CA^2, CA^3]$$
$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 3\omega^2 & 0 & 0 & 2\omega \\ 0 & -2\omega & 0 & 0 \\ 0 & -\omega^2 & 0 & 0 \\ -6\omega^3 & 0 & 0 & -4\omega^2 \end{bmatrix}$$

Observability

Rank of C is 4 and hence the above state-output system is observable. To minimize the measurement, we might be tempted to measure y_1 only not y_2. This gives

$$C_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix}^T$$

and

$$O_1 = [C_1, C_1A, C_1A^2, C_1A^3] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 3\omega^2 & 0 & 0 & 2\omega \\ 0 & -\omega^2 & 0 & 0 \end{bmatrix}$$

which is of rank 3.

Observability

However, if y_1 is not measured, we get

$$C_2 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \end{bmatrix}^T$$

$$O_2 = [C_2, C_2A, C_2A^2, C_2A^3]$$

$$= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -2\omega & 0 & 0 \\ -6\omega^3 & 0 & 0 & -4\omega^2 \end{bmatrix}$$

which is of rank 4. Thus, the state of the satellite is known from angle measurements alone but this is not so for radial measurements.

References

- Mohan C Joshi, *Ordinary Differential Equations - Modern Perspective*, Alpha Science International Limited, 2006.
- Roger W. Brockett, *Finite Dimensional Linear System*, John Wiley & Sons, 1970.