# Weierstrass Theorem and Some Generalizations

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# Introduction

Most calculus students are familiar with the concept of a Taylor polynomial and some of the associated results. While these polynomials can be made arbitrarily close to certain functions on a closed interval, they require that the functions be analytic (highly differentiable) which is a relatively small subclass of functions. This raises the question of whether this condition is actually necessary for a function to be approximated on a closed interval, to an arbitrary degree, by a polynomial. It turns out, as Karl Weierstrass proved with the Weierstrass Approximation Theorem, it is only necessary for $f$ to be continuous on the closed interval in order for such polynomials to exist.

# Introduction

Marshall Stone, with the Stone-Weierstrass theorem, generalized the result to any continuous function that maps the elements of a compact Hausdorff space into $\mathbb{R}$. This result also generalizes the approximating functions from polynomials to the members of a subalgebra of the continuous functions that map $K$ to $\mathbb{R}$.

The classical Weierstrass Approximation Theorem states that any continuous real-valued function defined on a bounded closed interval of real numbers can be approximated uniformly by polynomials. In this lecture, we discuss Weierstrass Approximation Theorem and some generalizations.

# Notations

| | |
|---|---|
| $\mathbb{R}$ | the set of real numbers. |
| $\mathbb{C}$ | the set of complex numbers. |
| $K$ | compact Hausdorff space. |
| $C(K,\mathbb{R})$ | the set of continuous functions from $K$ to $\mathbb{R}$. |
| $C(K,\mathbb{C})$ | the set of continuous functions from $K$ to $\mathbb{C}$. |
| $\mathcal{P}(K,\mathbb{R})$ | the space of polynomials from $K$ to $\mathbb{R}$ with real coefficients. |
| $\mathcal{P}(K,\mathbb{C})$ | the space of polynomials from $K$ to $\mathbb{C}$ with complex coefficients. |

# Brook **Taylor** : 1685 - 1731

**Brook Taylor** was an English mathematician best known for creating Taylor's theorem and the Taylor series, which are important for their use in mathematical analysis. Taylor series of a function is an infinite sum of terms that are expressed in terms of the function's derivatives at a single point. For most common functions, the function and the sum of its Taylor series are equal near this point.

The partial sum formed by the first $n+1$ terms of a Taylor series is a polynomial of degree $n$ that is called the $n$th Taylor polynomial of the function. Taylor polynomials are approximations of a function, which become generally better as $n$ increases[1].

[1] Source : Wikipedia

# Brook **Taylor** : 1685 - 1731

Taylor's theorem gives quantitative estimates on the error introduced by the use of such approximations.

# **Madhava** of Sangamagrama : 1350 - 1425

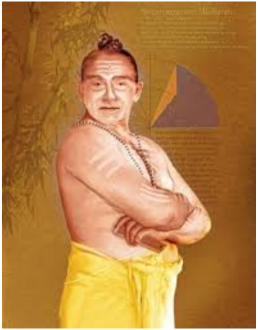
In the 14th century, the earliest examples of the use of Taylor series and closely related methods were given by **Madhava of Sangamagrama**.

Though no record of his work survives, writings of later Indian mathematicians suggest that he found a number of special cases of the Taylor series, including those for the trigonometric functions of sine, cosine, tangent, and arctangent[2].

[2] Source : Wikipedia

# **Madhava** of Sangamagrama : 1350 - 1425

The Kerala School of Astronomy and Mathematics further expanded his works with various series expansions and rational approximations until the 16th century.



# Karl **Weierstrass** : 1815 - 1897

**Karl Weierstrass** was a German mathematician often cited as the "**father of modern analysis**". Despite leaving university without a degree, he studied mathematics and trained as a school teacher, eventually teaching mathematics, physics, botany and gymnastics. He later received an honorary doctorate and became professor of mathematics in Berlin[3].

**Weierstrass Approximation Theorem**, one of the most important results in approximation theory is due Karl Weierstrass, proved in 1885[4] (when he was 70 years old!).

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[3] Source : Wikipedia

[4] K. Weierstrass (1885). Uber die analytische Darstellbarkeit sogenannter willkrlicher Functionen einer reellen Vernderlichen. Sitzungsberichte der Kniglich Preuischen Akademic der Wissenschaften zu Berlin, 1885 (11).

# Karl **Weierstrass** : 1815 - 1897

Among many other contributions, Weierstrass formalized the definition of the continuity of a function, proved the intermediate value theorem and the Bolzano-Weierstrass theorem, and used the latter to study the properties of continuous functions on closed bounded intervals[5].

[5]Source : https://mathshistory.st-andrews.ac.uk/

# Several Proofs

There are now several different proofs that use vastly different approaches. One well-known proof was given by Sergei Bernstein in 1911. His proof uses only elementary methods and gives an explicit algorithm for approximating a function by the use of a class of polynomials now bearing his name.

A constructive proof for Weierstrass Approximation Theorem was given by Bernstein in 1911 and is taken from the book by Davidson and Donsig[6].

[6] K. Davidson and A. Donsig. *Real Analysis with Real Applications.* Prentice Hall, Upper Saddle River, N.J., 2002.

# Sergei **Bernstein** : 1880 - 1968

**Sergei Bernstein** was a Ukranian mathematician who made important contributions to partial differential equations, differential geometry, probability theory and approximation theory[7].

[7]Source : https://mathshistory.st-andrews.ac.uk/

The Bernstein polynomials, which play a central role in Bernstein's proof, are introduced below.

**Definition 1 (Bernstein Polynomials).**

*For each $n \in \mathbb{N}$, the $n^{th}$ Bernstein polynomial of a function $f \in C([0,1], \mathbb{R})$ is defined as*

$$B_n(f)(x) := \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k}.$$

# Bernstein Theorem

## Theorem 2 (Bernstein Theorem).

*Let $f \in C([0,1],\mathbb{R})$. Then $B_n(f) \to f$ uniformly as $n \to \infty$.*

**Outline of the proof :**

1. Let $f \in C([0,1],\mathbb{R})$ and $\varepsilon > 0$. Since $f$ is continuous on $[0,1]$, $M : \|f\|_\infty = \{|f(x)| : x \in [0,1]\} < \infty$ ; $\exists \delta > 0$ such that $|x - y| < \delta \Rightarrow |f(x) - f(y)| < \frac{\varepsilon}{2}$, for all $x, y \in [0,1]$.
2. For each $\xi \in [0,1]$, $|B_n(f)(x) - f(\xi)| \leq \frac{2M(x-\xi)^2}{\delta^2} + \frac{2M(x-x^2)}{\delta^2} + \frac{\varepsilon}{2}$, for all $x \in [0,1]$.
3. In particular, $|B_n(f)(\xi) - f(\xi)| < \frac{\varepsilon}{2} + \frac{M}{2\delta^2 n}$.
4. Choose $N > \frac{M}{\delta^2 \varepsilon}$. Then $|B_n(f)(\xi) - f(\xi)| < \varepsilon$, for all $\xi \in [0,1]$.

## Remark 3.

*Note that the techinques used in the above proof can be modified to show that if $f$ is bounded on $[0,1]$ and $f$ is continuous at $x_0 \in [0,1]$, then the sequence $B_n(f)(x_0) \to f(x_0)$ as $n \to \infty$.*

**Theorem 4 (Weierstrass Approximation Theorem).**

*Let $f \in C([a, b], \mathbb{R})$. Then there is a sequence of ploynomials $p_n(x)$ that converges uniformly to $f(x)$ on $[a, b]$.*

**Outline of the proof :**

1. Let $f \in C([a, b], \mathbb{R})$. Consider the function $\phi : [0, 1] \to [a, b]$ defined by $\phi : x \mapsto a + (b - a)x$. The composite function $g := f \circ \phi$ is a continuous function on $[0, 1]$. By Bernstein theorem, for $\varepsilon > 0$, there exists a polynomial $q$ such that $\|q - g\|_\infty < \varepsilon$.
2. Define $\psi : [a, b] \to [0, 1]$ by $\psi(x) = \frac{x-a}{b-a}$, $\psi(x)$ is a polynomial. Let $p(x) = q(\frac{x-a}{b-a})$ and $p$ is a polynomial in $x$.
3. Since $\phi \circ \psi$ is the identity mapping on $[a, b]$, and $g = f \circ \phi$, we see that $g \circ \psi = f \circ \phi \circ \psi = f$.
4. Let $x \in [a, b]$. Then since $\psi(x) \in [0, 1]$, we have $|p(x) - f(x)| = |q(\psi(x) - g(\psi(x))| < \varepsilon$.

# Bernstein's Proof of Weierstrass Approximation Theorem

The above proof relied on only elementary methods.

The method of using Bernstein polynomials to prove the Weierstrass Theorem gives use a constructive method of finding a sequence of polynomials which converge uniformly on $[a, b]$ to the given continuous function.

# Weierstrass Approximation Theorem

One can easily prove the complex version of the Weierstrass Approximation Theorem. Note that $C([0,1],\mathbb{C})$ is a Banach space with respect to the supremum norm, $\|.\|_\infty$. We denote the set spanned by $S$ by $Span_{\mathbb{C}}\{S\}$.

**Theorem 5 (Weierstrass Approximation Theorem).**

*Let $f \in C([0,1]),\mathbb{C}$ and let $\varepsilon > 0$. Then, there exists a polynomial $p$ (with complex coefficients) such that*

$$\|f - p\|_\infty := \sup\{|f(x) - p(x)| : x \in [0,1]\} < \varepsilon$$

*In other words, $Span_{\mathbb{C}}\{x^n : n \in \mathbb{N} \cup \{0\}\}$ is dense in the Banach space $C([0,1],\mathbb{C})$ with the supremum norm, $\|.\|_\infty$.*

There is a natural question that **how many exponents are needed** in order to have density in $C([0,1],\mathbb{C})$. In 1912, at the Cambridge International Congress of Mathematicians, Bernstein posed this problem from the Weierstrass's result. Bernstein himself had worked on this problem, and he had proved that the condition

$$\sum_{k=1}^{\infty} \frac{1+\log \lambda_k}{\lambda_k} = \infty.$$

is necessary for density, and also that

$$\lim_{k\to\infty} \frac{\lambda_k}{k \log \lambda_k} = 0$$

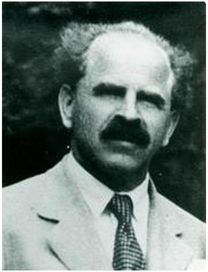
is sufficient.

# Herman **Müntz** (1884-1956) ; Otto **Szász** (1884-1952)

Two years later, in 1914, Benstein's conjecture was proven to hold by **Müntz**. The original proof of Müntz was simplified by **Szász** in 1916.

Müntz[8]

Szász[9]



[8] Source : Wikipedia

[9] Source : Wikipedia

# Muntz-Szasz Theorem

**Theorem 6 (Muntz-Szasz).**

*Let $\{\lambda_n\}_{n=1}^{\infty}$ be an increasing sequence of positive real numbers. Then,*

$$Span_{\mathbb{C}}\left\{x^{\lambda_n} : n \in \mathbb{N} \cup \{0\}\right\}$$

*is dense in $C([0,1], \mathbb{C})$ if and only if $\sum\limits_{n=1}^{\infty} \frac{1}{\lambda_n} = \infty$.*

# Muntz-Szasz Theorem

In 1916, Szasz published an article where he completed the proof, further improving and simplifying it. Muntz's demonstration uses real variable techniques and is based on estimating the distance between any continuous function to certain finite subspaces of polynomials, which can be made as small as desired. Szasz's proof makes use of complex variable techniques combined with some arguments of functional analysis. The proof we presented here can be found in Rudin's book[10] and followed Szasz's ideas.

[10] Walter Rudin. Principles of Mathematical Analysis, McGraw-Hill, Inc.

# Stone-Weierstrass Theorem

In 1937, Marshall H. Stone considerably generalized the theorem and then simplified the proof in 1948. His result is known as the Stone-Weierstrass Theorem. This theorem generalizes the Weierstrass Approximation Theorem in two ways:

1. Instead of the real interval $[a, b]$, an arbitrary compact Hausdorff space $K$ is considered, and
2. Instead of the algebra of polynomial functions, Stone investigated the approximation with elements from more general algebras of $C(K, \mathbb{K})$.

# Marshall H. Stone

It will be seen that the Weierstrass Approximation Theorem is in fact a special case of the more general Stone-Weierstrass Theorem, proved by Stone in 1937, who realized that very few of the properties of the polynomials were essential to the theorem. Although this proof is not constructive and relies on more machinery than that of Bernstein, it is much more efficient and has the added power of generality.

# Basic Concepts

The introduction of some new concepts and results is required before the proof of the Stone-Weierstrass Theorem can be approached.

## Definition 7 (unital sub-algebra, separating points).

*Let $K$ be a compact metric space. Consdider the Banach algebra*

$$C(K, \mathbb{R}) := \{f : K \to \mathbb{R} \mid f \text{ is continous}\}$$

*equipped with the sup-norm,* $\|f\|_\infty := \sup_{t \in K} |f(t)|$. *Then*

1. $A \subset C(K, \mathbb{R})$ *is a* ***unital sub-algebra*** *if* $1 \in A$ *and if* $f, g \in A, \alpha, \beta \in \mathbb{R}$ *implies that* $\alpha f + \beta g \in A$ *and* $fg \in A$.
2. $A \subset C(K, \mathbb{R})$ ***seperates points*** *of* $K$ *if for all* $s, t \in K$ *with* $s \neq t$, *there exists* $f \in A$ *such that* $f(s) \neq f(t)$.
3. *Furthermore, if, for all* $f, g \in C(K, \mathbb{R})$ *and* $x \in K$, *we define* $fg$ *by* $(fg)(x) = f(x)g(x)$, *it follows that* $fg \in C(K, \mathbb{R})$. *Thus, we see that* $C(K, \mathbb{R})$ *actually forms an algebra over* $\mathbb{R}$.

The above definition can also be generalized by replacing the requirement that $K$ be a compact metric space by requiring that $K$ be a compact topological space.

**Remark 8.**

(a) *It follows from the above definition that if $A$ is a unital sub-algebra, then all constant functions are elements of $A$.*

(b) *Let $\mathcal{P}([a,b],\mathbb{R})$ be the space of polynomials from $[a,b]$ to $\mathbb{R}$. It is easily checked that $\mathcal{P}([a,b],\mathbb{R})$ is a unital sub-algebra and separates points.*

# Basic Concepts

Before stating and proving the Stone-Weierstrass Theorem a useful lemma about closed sub-algebras will be proven. The following definition must first be made.

## Definition 9 (Lattice).

*A subset $S \in C(K, \mathbb{R})$ is a **lattice** if, for all $f, g \in S, f \vee g \in S$ and $f \wedge g \in S$, where $(f \vee g)(x) := \max\{f(x), g(x)\}$ and $(f \wedge g)(x) := \min\{f(x), g(x)\}$.*

## Lemma 10.

*Let $A \subset C(K, \mathbb{R})$ be a closed unital sub-algebra. Then*

i) *if $f \in A$ and $f \geq 0$, then $\sqrt{f} \in A$;*

ii) *if $f \in A$, then $|f| \in A$;*

iii) *$A$ is a lattice.*

We are now in a position to state and prove the Stone-Weierstrass theorem. The theorem was first proved by Stone in 1937[11]. However, he greatly simplified his proof in 1948 into the one that is commonly used today[12].

**Theorem 11 (Stone-Weierstrass Theorem).**

*Let $K$ be a compact metric space and $A \subset C(K, \mathbb{R})$ a unital sub-algebra which separates points of $K$. Then $A$ is dense in $C(K, \mathbb{R})$.*

[11] M. Stone (1937). Applications of the Theory of Boolean Rings to General Topology. Translations of the Americain Mathematical Society **41**(3), 375 – 481.

[12] M. Stone (1948). The Generalized Weierstrass Approximation Theorem. Mathematics Magazine **21**(21), 167 – 184 and **21**(5), 237 – 254.

**Remark 12.**

*An equivalent statement is that if A is a closed unital subalgebra that separates points of a compact set $K$, and $\mathcal{A} \subset C(K, \mathbb{R})$, then $\mathcal{A} = C(K, \mathbb{R})$. We will proceed using this formulation.*

**Corollary 13.**

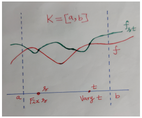
*Let $K$ be a compact subset of $\mathbb{R}^n$ for some $n \in \mathbb{N}$. Then the algebra of all polynomials $P(K, \mathbb{R})$ in the coordinates $x_1, x_2, \ldots, x_n$ is dense in $C(K, \mathbb{R})$.*

The case in which $n = 1$ in the above corollary is the Weierstrass Approximation Theorem.

# Proof of Stone-Weierstrass Theorem

**Outline of the proof :**

1. Let $f \in C(K, \mathbb{R})$ and $\varepsilon > 0$. Let $s, t \in K$ with $s \neq t$. Since $\mathcal{A}$ separates $K$, $\exists h \in \mathcal{A}$ such that $h(s) \neq h(t)$.
2. Define $f_{st}(x) = f(t) + \{f(s) - f(t)\}\frac{h(x)-h(t)}{h(s)-h(t)}$. As $\mathcal{A}$ is a closed unital subalgebra, $f_{st} \in \mathcal{A}$ with $f_{st}(s) = f(s)$ and $f_{st}(t) = f(t)$.
3. **Fix** $s \in K$ **and let** $t$ **vary.** Let $U_t = \{x \in K : f_{st}(x) < f(x) + \varepsilon\}$. Since $t \in U_t$ and $U_t$ is open, $\bigcup_{t\in K} U_t$ is an open cover of $K$. Hence by compactness of $K$, $K \subset \bigcup_{1\leqslant i\leqslant n} U_{t_i}$, for some $t_i \in K$, $1 \leq i \leq n$. Let $h_s := \min_{1\leqslant i\leqslant n} f_{st_i}$. So $h_s \in \mathcal{A}, h_s(s) = f(s)$ and $h_s < f + \varepsilon$.




4. Let $V_s = \{x \in K : h_s(x) > f(x) - \varepsilon\}$. Since $s \in V_s$ and $V_s$ is open, $\bigcup_{s\in K} V_s$ is an open cover of $K$. Hence by compactness of $K$, $K \subset \bigcup_{1\leqslant j\leqslant m} V_{s_j}$, for some $s_j \in K$, $1 \leq j \leq m$. Let $g := \max_{1\leqslant j\leqslant m} h_{s_j}$. So $g \in \mathcal{A}, g > f - \varepsilon$ and $g < f + \varepsilon$. Thus $\|f - g\|_\infty < \varepsilon$.

# Stone-Weierstrass Theorem

The Stone-Weierstrass Theorem can be used to prove the following two statements which go beyond Weierstrass' results.

1. If $f$ is a continuous real-valued function defined on the set $[a, b] \times [c, d]$ and $\varepsilon > 0$, then there exists a polynomial function $p$ in two variables such that $|f(x, y) - p(x, y)| < \varepsilon$ for all $x \in [a, b]$ and $y \in [c, d]$.
2. If $X$ and $Y$ are two compact Hausdorff spaces and $f : X \times Y \to \mathbb{R}$ is a continuous function, then for every $\varepsilon > 0$ there exist $n > 0$ and continuous functions $f_1, f_2, \ldots, f_n$ on $X$ and continuous functions $g_1, g_2, \ldots, g_n$ on $Y$ such that $\|f - \sum f_i g_i\|_{C([a,b]\times[c,d],\mathbb{R})} < \varepsilon$.

# Basic Results

**Theorem 14 (Stone-Weierstrass Theorem).**

*Suppose $\mathcal{A}$ is a subalgebra of $C(K, \mathbb{R})$ that separates points, where $K$ is a compact Hausdorff space. If there exists $x_0 \in X$ such that $f(x_0) = 0$ for all $f \in \mathcal{A}$, then $\mathcal{A}$ is dense in $\{f \in C(K, \mathbb{R}) : f(x_0) = 0\}$. Otherwise, $\mathcal{A}$ is dense in $C(K, \mathbb{R})$.*

**Theorem 15 (Stone-Weierstrass Theorem (Restatement)).**

*If $\mathcal{A}$ is a closed subalgebra of $C(K, \mathbb{R})$ that separates points, then either $\mathcal{A} = C(K, \mathbb{R})$ or $\mathcal{A} = \{f \in C(K, \mathbb{R}) : f(x_0) = 0\}$ for some $x_0 \in K$.*

# Basic Results

We shall now illustrate the Stone-Weierstrass Theorem in which $K$ consists of only two points, $x_1$ and $x_2$. Since any function $f : K \to \mathbb{R}$ is described entirely by the image of $x_1$ and $x_2$, each function can be represented by the ordered pair $(f(x_1), f(x_2))$. Thus, it will suffice to consider the closed subalgebras of $\mathbb{R}^2$.

## Lemma 16.

*Consider* $\mathbb{R}^2$ *as an algebra under coordinate addition and multiplication. The only subalgebras for* $\mathbb{R}^2$ *are* $\mathbb{R}^2$, $\{(0,0)\}$,$\{(x,0) : x \in \mathbb{R}\}$, $\{(0,x) : x \in \mathbb{R}\}$, *and* $\{(x,x) : x \in \mathbb{R}\}$.

**Proof**. Since each one of these sets is closed under coordinatewise addition and multiplication, they each form a subalgebra of $\mathbb{R}^2$. To see that these are the only ones, consider a point $(a,b) \in \mathcal{A}$. If $\mathcal{A}$ contains a point such that $a \neq b \neq 0$, then $(a,b)$ and $(a^2,b^2)$ are linearly independent. As a result, $\mathcal{A} = \mathbb{R}^2$. Now, the cases $a = b \neq 0$, $a \neq 0 = b$, or $a = 0 \neq b$ generate the other three nonzero subalgebras mentioned above. Finally, the only case remaining is if the only point happens when $a = b = 0$, which corresponds to the set $\{(0,0)\}$. Thus, the subalgebras mentioned above are the only possibilities.

# Proof of Stone-Weierstrass Theorem

Let $\mathcal{A}_{xy} = \{(f(x), f(y)) : f \in \mathcal{A}\}$. Now, since $\mathcal{A}$ is a subalgebra of $C(K, \mathbb{R})$, $\mathcal{A}_{xy}$ is a subalgebra of $\mathbb{R}^2$. Therefore, $\mathcal{A}_{xy}$ is either $\mathbb{R}^2$, $\{(0,0)\}, \{(x,0) : x \in \mathbb{R}\}$, $\{(0,x) : x \in \mathbb{R}\}$, or $\{(x,x) : x \in \mathbb{R}\}$. Now, since $\mathcal{A}$ separates points, $\mathcal{A}_{xy}$ cannot be $\{(0,0)\}$ or $\{(x,x) : x \in \mathbb{R}\}$. If $\mathcal{A}_{xy} = \mathbb{R}^2$, then $\mathcal{A} = C(K, \mathbb{R})$. Finally, if $\mathcal{A}_{xy}$ is $\{(x,0) : x \in \mathbb{R}\}$ or $\{(0,x) : x \in \mathbb{R}\}$, then there exists some $x_0 \in \mathbb{R}^2$ such that $f(x_0) = 0$ for all $f \in \mathcal{A}$. Furthermore, we have that $\mathcal{A} = \{f \in C(K, \mathbb{R}) : f(x_0) = 0\}$. Finally, note that if $\mathcal{A}$ contains a constant function, then there does not exist an $x_0$ such that $f(x_0) = 0$ for all $f \in \mathcal{A}$. Thus, $\mathcal{A} = C(K, \mathbb{R})$.

We can also use the Stone-Weierstrass Theorem to prove the complex Stone-Weierstrass Theorem. The complex version, however, requires additional assumptions. The following proof is taken from a book by Sohrab[13].

**Theorem 17 (Complex Stone-Weierstrass Theorem).**

*Let $A$ be a (complex) unital sub-algebra of $C(K, \mathbb{C})$, such that if $f \in A$, then $\bar{f} \in A$, and $A$ separates points of $K$. Then, $A$ is dense in $C(K, \mathbb{C})$.*

**Outline of the proof :** Let $f \in \mathcal{A}$. Then $Re(f) = \frac{f+\bar{f}}{2}$ and $Im(f) = \frac{f-\bar{f}}{2i}$ are in $\mathcal{A}$. Let $\mathcal{A}_{\mathbb{R}}$ denote the unital sub-algebra of $\mathcal{A}$ containing real-valued functions. Hence $\mathcal{A} = \{g + ih : g, h \in \mathcal{A}_{\mathbb{R}}\}$ is dense in $C(K, \mathbb{C}) = C(K, \mathbb{R}) + iC(K, \mathbb{R})$.

[13] H. Sohrab. *Basic Real Analysis*. Birkhauser, New York, N.Y., 2003.

# Trignometric Polynomials

A result similar to the Weierstrass Approximation Theorem occurs in the theory of Fourier series, and was also first proved by Weierstrass. It states that a continuous $2\pi$-periodic real-valued function can be uniformly approximated on $[a, b]$ by the trignometric polynomials. The space of all continuous $2\pi$-periodic real-valued functions is denoted by $C_{per}([0, 2\pi], \mathbb{R})$.

## Definition 18.

*The space of real-valued* ***trignometric polynomials*** $\mathcal{TP}(\mathbb{R}, \mathbb{R})$ *are functions* $f : \mathbb{R} \to \mathbb{R}$ *which are finite sums of the form*

$$f(x) = a_0 + \sum_{n=1}^{N} (a_n \cos(nx) + b_n \sin(nx)).$$

# Trignometric Polynomials

## Theorem 19.

*The set of all trignometric polynomials are uniformly dense in $C_{per}([0, 2\pi], \mathbb{R})$.*

**Outline of the proof :** We identity $C_{per}([0, 2\pi], \mathbb{R})$ with $C(\mathbb{T}, \mathbb{R})$, where $\mathbb{T} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ is the unit circle. Let $t \mapsto (\cos t, \sin t)$. Then the trigonometric polynomials in $x, y$ on $\mathbb{T}$, which are dense $C(\mathbb{T}, \mathbb{R})$.

Note that the above corollary can be easily generalized for $C_{per}([a, b], \mathbb{R})$.

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