Convergence of Taylor Series

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Overview

In this lecture, we address the following two questions.

1. When does a Taylor series converge to its generating function?
2. How accurately do a function's Taylor polynomials approximate the function on a given interval?

Taylor's Theorem

We answer the questions with the following theorem.

Theorem 1.

If f and its first n derivatives $f', f'', \ldots, f^{(n)}$ are continuous on the closed interval between a and b, and $f^{(n)}$ is differentiable on the open interval between a and b, then there exists a number c between a and b such that

$$f(b) = f(a) + f'(a)(b-a) + \frac{f''(a)}{2!}(b-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(b-a)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}.$$

Taylor's Theorem is a generalization of the Mean Value Theorem.

Proof of the theorem will be discussed at the end of this lecture.

Taylor's Formula

When we apply Taylor's Theorem, we usually want to hold "a" fixed and treat "b" as an independent variable. Taylor's formula is easier to use in circumstances like these if we change b to x. Here is a version of the theorem with this change.

Theorem 2.

If f has derivatives of all orders in an open interval I containing a, then for each positive integer n and for each x in I.

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x), \qquad (1)$$

where

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \qquad (2)$$

for some c between a and x.

Taylor's Formula

When we state Taylor's theorem this way, it says that for each $x \in I$,

$$f(x) = P_n(x) + R_n(x).$$

The function $R_n(x)$ is determined by the value of the $(n+1)$st derivative $f^{(n+1)}$ at a point c that depends on both a and x, and that lies somewhere between them.

For any value of n we want, the equation gives both a polynomial approximation of f of that order and a formula for the error involved in using that approximation over the interval I.

Taylor's Formula

Equation (1) is called **Taylor's formula**. The function $R_n(x)$ is called the **remainder of order n** or the **error term** for the approximation of f by $P_n(x)$ over I.

If $R_n(x) \to 0$ as $n \to \infty$ for all $x \in I$, we say that the Taylor series generated by f at $x = a$ **converges** to f on I, and we write

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

Often we can estimate R_n without knowing the value of c, as the following example illustrates.

Example 3.

Show that the Taylor series generated by $f(x) = e^x$ at $x = 0$ converges to $f(x)$ for every real value of x.

Solution. The function has derivatives of all orders throughout the interval $I = (-\infty, \infty)$. Equations (1) and (2) with $f(x) = e^x$ and $a = 0$ give

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + R_n(x)$$

and

$$R_n(x) = \frac{e^c}{(n+1)!}x^{n+1} \quad \text{for some } c \text{ between 0 and } x.$$

Since e^x is an increasing function of x, e^c lies between $e^0 = 1$ and e^x. When x is negative, so is c, and $e^c < 1$. When x is zero, $e^x = 1$ and $R_n(x) = 0$. When x is positive, so is c, and $e^c < e^x$.

Example

Thus, for $R_n(x)$ given as above,

$$|R_n(x)| \leq \frac{|x|^{n+1}}{(n+1)!} \quad \text{when } x \leq 0, \qquad e^c < 1$$

and

$$|R_n(x)| < e^x \frac{x^{n+1}}{(n+1)!} \quad \text{when } x > 0. \quad e^c < e^x$$

Finally, because

$$\lim_{n\to\infty} \frac{x^{n+1}}{(n+1)!} = 0 \quad \text{for every } x,$$

$\lim_{n\to\infty} R_n(x) = 0$, and the series converges to e^x for every x. Thus,

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^k}{k!} + \cdots. \tag{3}$$

Example

We can use the result of Example 3 with $x = 1$ to write

$$e = 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{n!} + R_n(1),$$

where for some c between 0 and 1,

$$R_n(1) = e^c \frac{1}{(n+1)!} < \frac{3}{(n+1)!} \qquad \textit{since} \quad e^c < e^1 < 3.$$

Estimating the Remainder

It is often possible to estimate $R_n(x)$ as we did in Example 3. This method of estimation is so convenient that we state it as a theorem for future reference.

Theorem 4 (The Remainder Estimation Theorem).

If there is a positive constant M such that $|f^{(n+1)}(t)| \leq M$ for all t between x and a, inclusive, then the remainder term $R_n(x)$ in Taylor's Theorem satisfies the inequality

$$|R_n(x)| \leq M \frac{|x-a|^{n+1}}{(n+1)!}.$$

If this inequality holds for every n and the other conditions of Taylor's Theorem are satisfied by f, then the series converges to $f(x)$.

We are now ready to look at some examples of how the Remainder Estimation Theorem and Taylor's Theorem can be used together to settle questions of convergence. As we will see, they can also be used to determine the accuracy with which a function is approximated by one of its Taylor polynomials.

Example 5.

Show that the Taylor series for $\sin x$ *at* $x = 0$ *converges for all* x*.*

Solution. The function and its derivatives are

$$\begin{aligned} f(x) &= \sin x, & f'(x) &= \cos x, \\ f''(x) &= -\sin x, & f'''(x) &= -\cos x, \\ &\vdots & &\vdots \\ f^{(2k)}(x) &= (-1)^k \sin x, & f^{(2k+1)}(x) &= (-1)^k \cos x, \end{aligned}$$

so

$$f^{(2k)}(0) = 0 \qquad \text{and} \qquad f^{(2k+1)}(0) = (-1)^k.$$

The series has only odd-powered terms and, for $n = 2k + 1$, Taylor's Theorem gives

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{(-1)^k x^{2k+1}}{(2k+1)!} + R_{2k+1}(x).$$

All the derivatives of $\sin x$ have absolute values less than or equal to 1, so we can apply the Remainder Estimation Theorem with $M = 1$ to obtain

$$|R_{2k+1}(x)| \leq 1 \cdot \frac{|x|^{2k+2}}{(2k+2)!}.$$

Since $(|x|^{2k+2}/(2k+2)!) \to 0$ as $k \to \infty$, whatever the value of x, so $R_{2k+1}(x) \to 0$ and the Maclaurin series for $\sin x$ converges to $\sin x$ for every x. Thus,

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots .$$

Example 6.

Show that the Taylor series for $\cos x$ at $x = 0$ converges to $\cos x$ for every value of x.

Solution. We add the remainder term to the Taylor polynomial for $\cos x$ to obtain Taylor's formula for $\cos x$ with $n = 2k$;

$$\cos\ x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^k \frac{x^{2k}}{(2k)!} \cdots + R_{2k}(x).$$

Because the derivatives of the cosine have absolute value less than or equal to 1, the Remainder Estimation Theorem with $M = 1$ gives

$$|R_{2k}(x)| \le 1.\frac{|x|^{2k+1}}{(2k+1)!}.$$

For every value of x, $R_{2k}(x) \to 0$ as $k \to \infty$. Therefore, the series converges to $\cos x$ for every value of x. Thus,

$$\cos\ x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots. \qquad (4)$$

Since every Taylor series is a power series, the operations of adding, subtracting, and multiplying Taylor series are all valid on the intersection of their intervals of convergence.

Example 7.

Using known series, find the first few terms of the Taylor series for the given function using power series operations.

(a) $\frac{1}{3}(2x + x \cos x)$

(b) $e^x \cos x$

Solution

$$(a) \quad \frac{1}{3}(2x + x \ \cos \ x) = \frac{2}{3}x + \frac{1}{3}x\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^k \frac{x^{2k}}{(2k)!} + \cdots\right)$$

$$= \frac{2}{3}x + \frac{1}{3}x - \frac{x^3}{3!} + \frac{x^5}{3.4!} - \cdots - x - \frac{x^3}{6} + \frac{x^5}{72} - \cdots$$

$$(b) \quad e^x \cos \ x = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right) . \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right)$$

$$= \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right) - \left(\frac{x^2}{2!} + \frac{x^3}{2!} + \frac{x^4}{2!2!} + \frac{x^5}{2!3!} \cdots\right)$$

$$+ \left(\frac{x^4}{4!} + \frac{x^5}{4!} + \frac{x^6}{2!4!} + \cdots\right) + \cdots$$

$$= 1 + x - \frac{x^3}{3} - \frac{x^4}{6} + \cdots$$

Using Taylor series

We recall that if $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely for $|x| < R$, then

$$\sum_{n=0}^{\infty} a_n (f(x))^n$$

converges absolutely for any continuous function f on $|f(x)| < R$.

We can use the Taylor series of the function f to find the Taylor series of $f(u(x))$ where $u(x)$ is any continuous function.

The Taylor series resulting from this substitution will converge for all x such that $u(x)$ lies within the interval of convergence of the Taylor series of f.

Using Taylor series

For instance, we can find the Taylor series for $\cos\ 2x$ by substituting $2x$ for x in the Taylor series for $\cos\ x$:

$$\begin{aligned}\cos\ 2x =& \sum_{k=0}^{\infty} \frac{(-1)^k (2x)^{2k}}{(2k)!} = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \cdots \\ =& 1 - \frac{2^2 x^2}{2!} + \frac{2^4 x^4}{4!} - \frac{2^6 x^6}{6!} + \cdots \\ =& \sum_{k=0}^{\infty} (-1)^k \frac{2^{2k} x^{2k}}{(2k)!}.\end{aligned}$$

Example 8.

For what values of x can we replace $\sin x$ by $x - \left(x^3/3!\right)$ with an error of magnitude no greater than 3×10^{-4}?

Solution. Here we can take advantage of the fact that the Taylor series for $\sin x$ is an alternating series for every nonzero value of x. According to the Alternating Series Estimation Theorem, the error in truncating

$$\sin\ x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

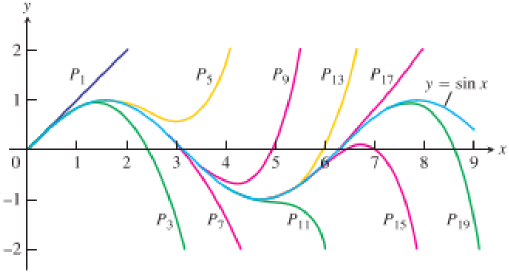
after $\left(x^3/3!\right)$ is no greater than

$$\left|\frac{x^5}{5!}\right| = \frac{|x|^5}{120}.$$

Therefore the error will be less than or equal to 3×10^{-4} if

$$\frac{|x|^5}{120} < 3 \times 10^{-4} \quad \text{or} \quad |x| < \sqrt[5]{306 \times 10^{-4}} \approx 0.514.$$

The Alternating Series Estimation Theorem tells us something that the Remainder Estimation Theorem does not: namely, that the estimate $x - \left(x^3/3!\right)$ for $\sin x$ is an underestimate when x is positive, because then $x^5/120$ is positive.



The polynomials

$$P_{2n+1}(x) = \sum_{k=0}^{n} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

converge to $\sin x$ as $n \to \infty$. Notice how closely $P_3(x)$ approximates the sine curve for $x \leq 1$.

The above figure shows the graph of $\sin x$, along with the graphs of a number of its approximating Taylor polynomials.

The graph of $P_3(x) = x - \left(x^3/3!\right)$ is almost indistinguishable from the sine curve when $0 \leq x \leq 1$.

Proof of Taylor's Theorem

We prove Taylor's theorem assuming $a < b$. The proof for $a > b$ is nearly the same. The Taylor polynomial

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

and its first n derivatives match the function f and its first n derivatives at $x = a$. We do not disturb that matching if we add another term of the form $K(x-a)^{n+1}$, where K is any constant, because such a term and its first n derivatives are all equal to zero at $x = a$. The new function

$$\phi_n(x) = P_n(x) + K(x-a)^{n+1}$$

and its first n derivatives still agree with f and its first n derivatives at $x = a$.

Proof of Taylor's Theorem (contd...)

We now choose the particular value of K that makes the curve $y = \phi_n(x)$ agree with the original curve $y = f(x)$ at $x = b$. In symbols,

$$f(b) = P_n(b) + K(b-a)^{n+1}, \quad or \quad K = \frac{f(b) - P_n(b)}{(b-a)^{n+1}}. \tag{5}$$

With K defined by Equation (5), the function

$$F(x) = f(x) - \phi_n(x)$$

measures the difference between the original function f and the approximating function ϕ_n for each x in $[a, b]$.

Proof of Taylor's Theorem (contd...)

We now use Rolle's Theorem. First, because $F(a) = F(b) = 0$ and both F and F' are continuous on $[a, b]$, we know that

$$F'(c_1) = 0$$

for some c_1 in (a, b). Next, because $F'(a) = F'(c_1) = 0$ and both F' and F'' are continuous on $[a, c_1]$ we know that

$$F''(c_2) = 0$$

for some c_2 in (a, c_1). Rolle's Theorem, applied successively to $F'', F''', \ldots, F^{(n-1)}$ implies the existence of

$$\begin{aligned}
&c_3 \quad \text{in}\,(a, c_2) \quad \text{such that} \quad F'''(c_3) = 0,\\
&c_4 \quad \text{in}\,(a, c_3) \quad \text{such that} \quad F^{(4)}(c_4) = 0,\\
&\vdots\\
&c_n \quad \text{in}\,(a, c_{n-1}) \quad \text{such that} \quad F^{(n)}(c_n) = 0.
\end{aligned}$$

Proof of Taylor's Theorem (contd...)

Finally, because $F^{(n)}$ is continuous on $[a, c_n]$ and differentiable on (a, c_n), and $F^{(n)}(a) = F^{(n)}(c_n) = 0$, Rolle's Theorem implies that there is a number c_{n+1} in (a, c_n) such that

$$F^{(n+1)}(c_{n+1}) = 0 \tag{6}$$

If we differentiate $F(x) = f(x) - P_n(x) - K(x-a)^{n+1}$ a total of $n+1$ times, we get

$$F^{(n+1)}(x) = f^{(n+1)}(x) - 0 - (n+1)!K. \tag{7}$$

Proof of Taylor's Theorem (contd...)

Equations (6) and (7) together give

$$K = \frac{f^{(n+1)}(c)}{(n+1)!} \quad \text{for some number} \quad c = c_{n+1} \text{ in } (a, b)\,. \tag{8}$$

Equation (5) and (8) give

$$f(b) = P_n(b) + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}\,.$$

This concludes the proof.

Exercises 9.

Use substitution to find the Taylor series at $x = 0$ of the functions in the following exercises.

1. $\sin\left(\frac{\pi x}{2}\right)$
2. $\cos\left(x^{2/3}/\sqrt{2}\right)$
3. $\tan^{-1}(3x^4)$
4. $\frac{1}{1+\frac{3}{4}x^3}$

Exercises 10.

Use power series operations to find the Taylor series at $x = 0$ for the functions in the following exercises.

1. xe^x
2. $\frac{x^2}{2} - 1 + \cos x$
3. $x \ln(1 + 2x)$
4. $\sin x \cdot \cos x$
5. $\cos x - \sin x$
6. $\ln(1 + x) - \ln(1 - x)$

Exercises 11.

Find the first four nonzero terms in the Maclaurin series for the functions in the following exercises.

1. $\frac{\ln(1+x)}{1-x}$
2. $(\tan^{-1} x)^2$
3. $\cos^2 x \cdot \sin x$
4. $\sin(\tan^{-1} x)$

Exercises 12.

1. *Estimate the error if* $P_3(x) = x - (x^3/6)$ *is used to estimate the value of* $\sin x$ *at* $x = 0.1$.
2. *Estimate the error if* $P_4(x) = 1 + x + (x^2/2) + (x^3/6) + (x^4/24)$ *is used to estimate the value of* e^x *at* $x = 1/2$.
3. *For approximately what values of* x *can you replace* $\sin x$ *by* $x - (x^3/6)$ *with an error of magnitude no greater than* 5×10^{-4}*? Give reasons for your answer.*
4. *If* $\cos x$ *is replaced by* $1 - (x^2/2)$ *and* $|x| < 0.5$, *what estimate can be made of the error? Does* $1 - (x^2/2)$ *tend to be too large, or too small? Give reasons for your answer.*

Exercises 13.

1. *How close is the approximation* $\sin x = x$ *when* $|x| < 10^{-3}$*? For which of these values of* x *is* $x < \sin x$*?*
2. *The estimate* $\sqrt{1+x} = 1 + (x/2)$ *is used when* x *is small. Estimate the error when* $|x| < 0.01$.
3. *The approximation* $e^x = 1 + x + (x^2/2)$ *is used when* x *is small. Use the Remainder Estimation Theorem to estimate the error when* $|x| < 0.1$.
4. *(Continuation of the above exercise) When* $x < 0$*, the series for* e^x *is an alternating series. Use the Alternating Series Estimation Theorem to estimate the error that results from replacing* e^x *by* $1 + x + (x^2/2)$ *when* $-0.1 < x < 0$*. Compare your estimate with the one you obtained in the above exercise.*

Exercises 14.

1. *Use the identity* $\sin^2 x = (1 - \cos 2x)/2$ *to obtain the Maclaurin series for* $\sin^2 x$. *Then differentiate this series to obtain the Maclaurin series for* $2 \sin x \cos x$. *Check that this is the series for* $\sin 2x$.
2. *(Continuation of the above xercise.) Use the identity* $\cos^2 x = \cos 2x + \sin^2 x$ *to obtain a power series for* $\cos^2 x$.
3. **Taylor's Theorem and the Mean Value Theorem.** *Explain how the Mean Value Theorem is a special case of Taylor's Theorem.*
4. **Linearizations at inflection points.** *Show that if the graph of a twice-differentiable function* $f(x)$ *has an inflection point at* $x = a$, *then the linearization of* f *at* $x = a$ *is also the quadratic approximation of* f *at* $x = a$. *This explains why tangent lines fit so well at inflection points.*

Exercises 15.

1. **The (second) second derivative test.** *Use the equation*

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(c_2)}{2}(x-a)^2$$

to establish the following test:
Let f have continuous first and second derivatives and suppose that $f'(a) = 0$*. Then*

   a. *f has a local maximum at a if* $f'' \leq 0$ *throughout an interval whose interior contains a;*
   b. *f has a local maximum at a if* $f'' \geq 0$ *throughout an interval whose interior contains a.*

Exercises 16.

1. **A cubic approximation.** *Use Taylor's formula with* $a = 0$ *and* $n = 3$ *to find the standard cubic approximation of* $f(x) = 1/(1-x)$ *at* $x = 0$. *Give an upper bound for the magnitude of the error in the approximation when* $|x| \leq 0.1$.
2. a. *Use Taylor's formula with* $n = 2$ *to find the quadratic approximation of* $f(x) = (1+x)^k$ *at* $x = 0$ *(*k *a constant).*

   b. *If* $k = 3$, *for approximately what values of* x *in the interval* $[0, 1]$ *will the error in the quadratic approximation be less than* $1/100$*?*
3. **Improving approximations of** π**:**

   a. *Let* P *be an approximation of* π *accurate to* n *decimals. Show that* $P + \sin P$ *gives an approximation correct to* $3n$ *decimals. (Hint : Let* $P = \pi + x$.*)*

   b. *Try it with a calculator.*

Exercise 17.

A function defined by a power series $\sum_{n=0}^{\infty} a_n x^n$ *with a radius of convergence* $R > 0$ *has a Taylor series that converges to the function at every point of* $(-R, R)$*. Show this by showing that the Taylor series generated by* $f(x) = \sum_{n=0}^{\infty} a_n x^n$ *is the series* $\sum_{n=0}^{\infty} a_n x^n$ *itself.*
An immediate consequence of this is that series like

$$x \sin x = x^2 - \frac{x^4}{3!} + \frac{x^6}{5!} - \frac{x^8}{7!} + \dots$$

and

$$x^2 e^x = x^2 + x^3 + \frac{x^4}{2!} + \frac{x^5}{3!} + \dots,$$

obtained by multiplying Taylor series by powers of x*, as well as series obtained by integration and differentiation of convergent power series, are themselves the Taylor series generated by the functions they represent.*

Exercises 18.

1. **Taylor series for even functions and odd functions.** *Suppose that* $f(x) = \sum_{n=0}^{\infty} a_n x^n$ *converges for all* x *in an open interval* $(-R, R)$. *Show that*
   a. *If* f *is even, then* $a_1 = a_3 = a_5 = \cdots = 0$, *i.e., the Taylor series for* f *at* $x = 0$ *contains only even powers of* x.
   b. *If* f *is odd, then* $a_0 = a_2 = a_4 = \cdots = 0$, *i.e., the Taylor series for* f *at* $x = 0$ *contains only odd powers of* x.

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