Department of Mathematical and Computational Sciences
National Institute of Technology Karnataka, Surathkal

sam@nitk.edu.in https://sam.nitk.ac.in/

MA111 - Engineering Mathematics - II
Problem Sheet - 1

Sequences

1. Find the limit of the following sequences whose nth term is given by the formula
(i) $\dfrac{(-1)^n}{n+1}$ (ii) $\dfrac{2n}{3n^2+1}$ (iii) $\dfrac{2n^2+3}{3n^2+1}$

   (**Ans**: (i) 0, (ii) 0, (iii) 2/3).

2. Show that the sequences given in question 1 converges to the corresponding limits by $\epsilon - N$ definition.

3. Discuss the convergence of the sequence (a_n) defined recursively by (i) $a_1 = 1$, $a_{n+1} = 2 - 3a_n,\ n = 1, 2, \ldots$ (ii) $a_1 = 1$ and $a_{n+1} = \frac{a_n}{1+a_n},\ n = 1, 2, \ldots$.

   (**Ans**: (i) divergent (ii) convergent)

4. Let $a_1 = 2$, $a_{n+1} = \frac{1}{2}(a_n + \frac{2}{a_n}),\ n = 1, 2, \ldots$ Show that (a_n) is decreasing and bounded below by $\sqrt{2}$.

5. Find the limit of the sequence

$$\left\{\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \ldots\right\}.$$

   Ans: 2.

6. Find the limit of (i) $a_n = \left(1 + \dfrac{1}{n}\right)^n$ (ii) $a_n = \left(\dfrac{3n+1}{3n-1}\right)^{1/n}$.

   (**Ans**: (i) e, (ii) $e^{2/3}$).

7. For any real number x, show that $\left(\dfrac{x^n}{n!}\right)$ converges.

8. Show that $\left(\frac{\log n}{n^c}\right) \to 0$ for any $c > 0$.

9. Give an example of a continuous function $f(x)$ and a sequence (a_n) such that $f(a_n)$ converges but (a_n) diverges.

10. Discuss the convergence of
(i) $\frac{\sin^2 n}{2n}$ (ii) $\frac{n!}{2^n 3^n}$ (iii) $\frac{n!}{n^n}$ (iv) $n^{1/n}$ (v) $\sqrt{n} - \sqrt{n+1}$.

    (**Ans**: (ii) divergent. (i),(ii),(iv),(v) convergent.)

11. Give an example of a sequence (a_n) of positive numbers which converges but the sequence (b_n) diverges where $b_n = \frac{a_{n+1}}{a_n}$.

12. Prove that if $\{a_n\}$ is a convergent sequence, then to every positive number ϵ there corresponds an integer N such that for all m and n, $m > N$ and $n > N \Rightarrow |a_m - a_n| < \epsilon$.

13. Let $a_1 = a, a_2 = f(a_1), a_3 = f(a_2) = f(f(a)), \ldots, a_{n+1} = f(a_n)$, where f is a continuous function. If $\lim_{n\to\infty} a_n = L$, show that $f(L) = L$.

14. Prove that if a sequence (a_n) converges to a limit L, then every subsequence of (a_n) also converges to L.

15. For a sequence (a_n) the terms of even index are denoted by a_{2k} and the terms of odd index by a_{2k-1}. Prove that if $a_{2k} \to L$ and $a_{2k-1} \to L$, then $a_n \to L$.

16. Define the sequences $\{a_n\}$ and $\{b_n\}$ as follows:

$$0 < b_1 < a_1, a_{n+1} = \frac{a_n + b_n}{2} \text{ and } b_{n+1} = \sqrt{a_n b_n} \text{ for } n \in \mathbb{N}.$$

Show that $\{a_n\}$ and $\{b_n\}$ both tend to the same limit. This limit is called the arithmetic-geometric mean of a_1 and b_1.

17. Let the sequence (a_n) be defined by $a_n = \lim\limits_{n\to\infty} \frac{[x]+[2x]+\ldots+[nx]}{n^2}$, where x is a real number. Is this sequence convergent? If so, what is the limit? (**Ans**: $x/2$)

18. Show that the sequence $\{(1 + 1/n)^n\}$ is a monotone increasing sequence, bounded above.

19. Let $\{b_n\}$ be a bounded sequence which satisfies the condition $b_{n+1} \geq b_n - \frac{1}{2^n}, n \in \mathbb{N}$. Show that the sequence $\{b_n\}$ is convergent.

20. For $c > 2$, the sequence $\{p_n\}$ is defined recursively by $p_1 = c^2, p_{n+1} = (p_n - c)^2, n > 1$. Show that the sequence (p_n) strictly increases.

    [Hint. By induction, first prove that $p_n > 2c$.]

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