Recurrence Relations and Their Solutions (Problem : Steiner's Regions of Space)

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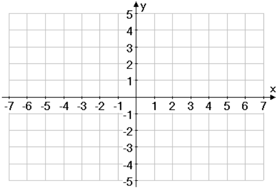
Lines in the Plane

Consider the plane $\mathbb{R} \times \mathbb{R}$.



Problem : What is the **maximum number** L_n of regions defined by n lines in the plane?

Here each line extends infinitely in both directions. This problem was first solved in 1826, by the Swiss mathematician **Jacob Steiner**.

Jacob Steiner (1796-1863)

Convexity of a set

Again we start by looking at small cases. The plane with

- no lines, has **one** region,
- one line, has **two** regions,
- two parallel lines contribute **three regions** but two non-parallel lines contribute **four** regions.

Definition

A region is convex *if it includes all line segments between* any two *of its points.*

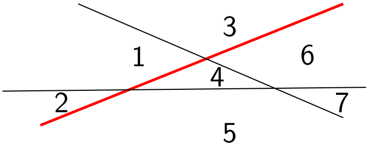
Observations :

- The given plane, $\mathbb{R} \times \mathbb{R}$ is convex.
- Any region, splitted by any number of straight lines, is convex.
- A straight line can split a convex region into at most two new regions, which are also convex.

What happens if there are 3 or more lines ?

Case : When there are 3 lines

When we add the third line, it can split at most 3 of the old regions, no matter how the first two lines are placed. In this case, $L_3 = 7$.



Case : When there are n lines

To attain the maximum number of regions, we should have the following assumptions.

1. no two lines should be parallel, and
2. no three lines should be concurrent.

What happens if there are n lines ?

Exercise

1. *Verify that the following are equivalent :*
   (a) *The nth line (for $n \geq 0$) increases the number of regions by k.*
   (b) *The nth line splits k of the old regions.*
   (c) *The nth line hits the previous lines in $k - 1$ different places.*

Since nth line can intersect the "$n-1$ old lines" in at most "$n-1$" different points, k is at most n. That is, $k \leq n$.

Therefore we have established the upper bound

$$L_n \leq L_{n-1} + n, \qquad \text{for} \quad n > 0.$$

We can achieve equality with the following the assumptions:

- nth line is not parallel to any of the others
- nth line does not go through any of the existing intersection points.

Recurrence Relation

The recurrence relation is therefore given by

$$\begin{aligned} L_0 &= 1 \\ L_n &= L_{n-1} + n, \qquad \text{for} \quad n > 0. \end{aligned}$$

We should look the solution at small cases :

n	0	1	2	3	4	5
L_n	1	2	4	7	11	16

It looks as if $L_n = 1 + S_n$, where S_n is the sum of first n positive integers.

We can often understand a recurrence by "unfolding" or "unwinding" it all the way to the end, as follows:

$$\begin{aligned} L_n &= L_{n-1} + n \\ &= L_{n-2} + (n-1) + n \\ &\vdots \\ &= L_0 + 1 + 2 + \cdots + n \end{aligned}$$

Triangular Numbers

Thus $L_n = 1 + S_n$. The values of S_n, $(1, 3, 6, 10, 15, \ldots)$ are called the **triangular numbers**, because S_n is the number of bowling pins in an n-row triangular array.

$\leftarrow$ the usual 4-row array

Adding S_n to its reversal gives that $2S_n = n(n+1)$.

Thanks to Gauss who came up with the trick in 1786 when he was nine years old. Thus

$$L_n = 1 + \frac{n(n+1)}{2}, \qquad \text{for} \quad n \geq 0. \tag{1}$$

Closed Form

Exercise

2. *Prove (1) by induction.*

Non-closed Form	**Closed Form**
$T_0 = 0$ $T_n = 2T_{n-1} + 1, \quad n > 0$	$T_n = 2^n - 1, \quad n \geq 0$
$L_0 = 1$ $L_n = L_{n-1} + n, \quad n > 0$	$L_n = 1 + \frac{n(n+1)}{2}, \quad n \geq 0$
$S_n = 1 + 2 + \cdots + n, \quad n > 0$	$S_n = \frac{n(n+1)}{2}, \quad n > 0$

Definition

An expression for a quantity $f(n)$ is in closed form *if we can compute it using at most a fixed number of "well known" standard operations, independent of n.*

Closed Form

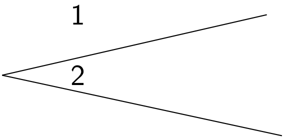
For example, $2^n - 1$ and $n(n+1)/2$ are closed forms, because they involve only addition, subtraction, multiplication, division and exponentiation, in explicit ways.

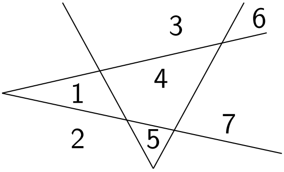
- The total number of simple closed forms is limited.
- There are recurrences that don't have simple closed forms.
- The product of first n integers $n!$ has proved to be so important that we now consider it a basic operation. The formula "$n!$" is therefore in closed form, although its equivalent "$1 \cdot 2 \cdots n$" is not.

A Variant of the "Lines-in-the-Plane" Problem

Suppose that instead of straight lines we use bent lines (∨-shaped lines, called ∨-lines) each containing one "zig".

What is the maximum number Z_n of regions determined by 'n' ∨-lines in the planes?





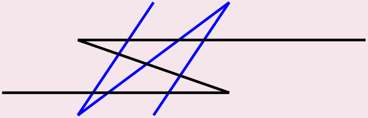
From the above two cases, we realize that ∨-line is like "two straight lines":

- at intersection (zig point) of a ∨-line, if the lines are extended, we would get 4 regions,
- we lose two regions when there is a ∨-line.

In order to get the maximum regions, we assume that the intersecting point of nth $\vee$-line does not lie on the existing intersection point of other "$n-1$" $\vee$-lines. There are "n" $\vee$-lines and we lose only two regions per line. Thus $Z_n = L_{2n} - 2n = 2n^2 - n + 1$, for $n \geq 0$.

Exercises

3. *For large value of n, can we say that there are four times as many regions with bent lines as with straight lines?*
4. *Some of the regions defined by n lines in the plane are infinite, while others are bounded. What is the maximum possible number of bounded regions?*
5. *What is the maximum number of regions definable by n zig-zag lines, each of which consists of two parallel infinite half-lines joined by a straight segment?*



Planes in the Space

We consider the problem of determining the maximum number of $3D$ regions that can be defined by n planes in space. We denote the **maximum number** of $3D$ regions by P_n. To attain the maximum, we assume the following :

- no two planes should be parallel (because we can inrease the number of regions by tilting one of the planes),
- meets of a plane with two others are never parallel lines (Suppose two planes are meeting the third plane in two parallel lines. If we move one of the two planes so that there are two crossing lines in the third plane, then the number of regions will increase), and
- no more than 3 planes meet at a point (Three planes always meet at a point because the two planes meet at a line, one more plane should meet the line at a point. But fourth plane should not go through the common intersection of three existing planes. Otherwise, we can move the fourth plane little bit, there is an increase in number of regions.)

The solution at small cases is given below.

n	0	1	2	3
P_n	1	2	4	8

Do the four planes divide the space into 16 parts ?

The $(n+1)$th plane meets each of the existing n planes in a line. Morover,

- no two of which are parallel, and
- no three of them going to meet.

Now we can look the $(n+1)$th plane, which consists n lines. Each line in the $(n+1)$-plane is the intersection of $(n+1)$th plane with each of the existing n planes.

Hence the problem is reduced to a situation of finding maximum number of regions generated by n-lines, which produces L_n regions.

Each of the region in the $(n+1)$th plane will slice the existing regions formed from n planes into 2. Each region contributes a new region in space that was not existing before. Hence

$$P_{n+1} = P_n + L_n.$$

The above argument can be extended to m-dimensional space. We can find maximum number of regions generated by m-planes (called hyperplanes, having dimension $m-1$). To get maximum number of regions, we should avoid parallel planes, lines and multiple intersections.

Exercises

6. *How many pieces of cheese can you obtain from a single thick piece by making five straight slices? (The cheese must stay in its original position while you do all the cutting, and each slice must correspond to a plane in 3D.) Find a recurrence relation for P_n, the maximum number of three-dimensional regions that can be defined by n different planes. [Answer : $P_n = \frac{n^3+5n+6}{6}$]*
7. *Show that the following set of n bent lines defines Z_n regions, where Z_n is defined by*

$$Z_n = 2n^2 - n + 1, \quad \textit{for} \quad n \geq 0.$$

The jth bent line, for $1 \leq j \leq n$, has its zig at $(n^{2j}, 0)$ and goes up through the points $(n^{2j} - n^j, 1)$ and $(n^{2j} - n^j - n^{-n}, 1)$.

8. *Is it possible to obtain Z_n regions with n bent lines when the angle at each zig is 30°?*

References

1. **Graham, Knuth and Patashnik**, *"Concrete Mathematics – A Foundation for Computer Science"*, Pearson Education.
2. **Marko Petkovsek, Herbert S. Wilf and Doron Zeilberger**, "$A = B$", AK Peters Ltd., Wellesley, Massachusetts.
3. **Herbert S. Wilf**, *"Generatingfunctionology"*, Third Edition, AK Peters Ltd., Wellesley, Massachusetts.