# Generating Function

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# Overview

The most powerful way to deal with sequences of numbers is to manipulate infinite series that "generate" those sequences.

A function (series) generated by the sequence is called generating function.

An important use of generating functions is to solve recurrence relations.

# Generating Function

Given an infinite sequence $\langle a_0, a_1, a_2, \ldots \rangle$, we have a power series in an auxilary variable $z$,

$$A(z) = a_0 + a_1 z + a_2 z^2 + \cdots = \sum_{k \geq 0} a_k z^k.$$

$A(z)$ is called a **generating function**. It is a single quantity which represents an entire infinite sequence.

The sequences $\langle \frac{1}{n!} \rangle, \langle 1 \rangle, \langle 1, \alpha, \alpha^2, \ldots \rangle$ have $e^z$, $\frac{1}{1-z}$ and $\frac{1}{1-\alpha z}$ as their generating functions respectively.

# Convolution of sequences

If $A(z)$ is any power series $\sum_{k\geq 0} a_k z^k$ ($A(z)$ is the generating function for $\langle a_0, a_1, a_2, \ldots \rangle$), then the coefficient of $z^n$ in $A(z)$ is denoted by

$$[z]^n A(z).$$

Let $A(z)$ and $B(z)$ be the generating functions for $\langle a_0, a_1, a_2, \ldots \rangle$ and $\langle b_0, b_1, b_2, \ldots \rangle$ respectively.

Then the product $A(z)B(z)$ (denoted by $c_n$) is

$$c_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0 = \sum_{k=0}^{n} a_k b_{n-k}.$$

The sequence $\langle c_n \rangle$ is called the **convolution** of the sequences $\langle a_n \rangle$ and $\langle b_n \rangle$.

# Some useful identities

- Two sequences are "convolved" by forming the sums of all products, whose subscripts add up to a given amount.
- Convolution of sequences corresponds to multiplication of their generating functions.

Generating functions give us powerful ways to discover and / or prove identities: The binomial theorem gives the following:

$$(1+z)^r = \sum_{k\geq 0} \binom{r}{k} z^k$$

and

$$(1+z)^s = \sum_{k\geq 0} \binom{s}{k} z^k.$$

If we multiply these together, we get another generating function:

$$(1+z)^r(1+z)^r = (1+z)^{r+s}.$$

# Some useful identities

Equating coefficients of $z^n$ on both sides of this equation gives us

$$\sum_{k\geq 0} \binom{r}{k}\binom{s}{n-k} = \binom{r+s}{n}.$$

We know that the generating function for the sequence

$$\langle (-1)^n \binom{r}{n} \rangle = \langle \binom{r}{0}, -\binom{r}{1}, \binom{r}{2}, \ldots \rangle$$

is $(1-z)^r$. Equating coefficients of $z^n$ from

$$(1-z)^r(1+z)^r = (1-z^2)^r$$

gives the equation

$$\sum_{k\geq 0} \binom{n}{k}\binom{r}{n-k}(-1)^k = (-1)^{n/2}\binom{r}{n/2}[n \text{ even}].$$

# Some important identities

The following important identities arise so frequently in applications.

$$\frac{1}{(1-z)^{n+1}} = \sum_{k\geq 0} \binom{n+k}{n} z^k, \quad \text{integer} \quad n \geq 0 \tag{1}$$

$$\frac{z^n}{(1-z)^{n+1}} = \sum_{k\geq 0} \binom{k}{n} z^k, \quad \text{integer} \quad n \geq 0. \tag{2}$$

(21) comes from

$$(1-z)^{-(n+1)} = \sum_{k\geq 0} \binom{-n-1}{k} (-1)^k z^k = \sum_{k\geq 0} \binom{k+n}{k} z^k.$$

(2) comes from (21) multiplied by $z^n$, that is, "shifted right" by $n$ places.

When $n = 0$, we get the geometric series

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots = \sum_{k \geq 0} z^k.$$

This is the generating function for the sequence $\langle 1, 1, \ldots \rangle$, and it is especially useful because the convolution of any other sequence $\langle b_0, b_1, \ldots \rangle$ with $\langle 1, 1, \ldots \rangle$ is the sequence of sums

$$\langle b_0, b_0 + b_1, b_0 + b_1 + b_2, \ldots \rangle.$$

That is, if $B(z)$ is the generating function for the sequence $\langle b_0, b_1, \ldots \rangle$, then $B(z)/(1-z)$ is the generating function for the sequence $\langle b_0, b_0 + b_1, b_0 + b_1 + b_2, \ldots \rangle$.

**Exercise 1.**

1. *If $A(z)$ is the generating function for the sequence $\langle a_0, a_1, a_2, \ldots \rangle$, then find the sequence corresponding to $A(z)(1-z)$.*

# Reshaping "generating functions" by adding / shifting some generating functions.

We can view the generating function $G(z)$, a function of a complex variable $z$.

We can reshape generating functions by adding, shifting, changing variable, differentiating, integrating, and multiplying generating functions.

$$\begin{aligned}
\alpha F(z) + \beta G(z) &= \sum_n (\alpha f_n + \beta g_n) z^n \\
z^m G(z) &= \sum_n g_{n-m} z^n, \quad \text{integer } m \geq 0 \\
G(cz) &= \sum_n c^n g_n z^n
\end{aligned}$$

## Reshaping "generating functions" changing variable, differentiating, integrating, and multiplying some generating functions.

$$
\begin{aligned}
G'(z) &= \sum_{n} n g_n z^n \\
\int_0^z G(t)\,dt &= \sum_{n\geq 1} \frac{1}{n} g_{n-1} z^n \\
F(z)G(z) &= \sum_{n} \Big( \sum_{k} f_k g_{n-k} \Big) z^n \\
\frac{G(z) - g_0 - g_1 z - \cdots - g_{m-1} z^{m-1}}{z^m} &= \sum_{n\geq 0} g_{n+m} z^n, \quad \text{integer } m \geq 0 \\
\frac{1}{1-z} G(z) &= \sum_{n} \Big( \sum_{k\geq n} g_k \Big) z^n.
\end{aligned}
$$

# Closed forms

Two kinds of "closed forms" come up when we work with generating functions. We observed that a generating function

$$G(z) = \sum_{n \geq 0} g_n z^n$$

has a sequence $\langle g_0, g_1, g_2, \ldots \rangle$ and vice-versa.

We have a closed form for $G(z)$, expressed in terms of $z$, or we have a closed form for $g_n$, expressed in terms of $n$.

Each of the generating functions in the following table is important enough to be remembered.

Many of them are special cases of the others and many of them can be derived quicky from the others by using the basic operations.

For example, let us consider the sequence $\langle 1, 2, 3, \ldots \rangle$ whose generating function $\frac{1}{(1-z)^2}$ is often useful.

It is the special case $m = 1$ of

$$\left\langle 1, \binom{m+1}{m}, \binom{m+2}{m}, \binom{m+3}{m}, \ldots \right\rangle ;$$

it is also the special case $c = 2$ of

$$\left\langle 1, c, \binom{c+1}{2}, \binom{c+2}{3}, \ldots \right\rangle .$$

We can derive it from the generating function for $\langle 1, 1, 1, \ldots \rangle$ by taking cumulative sums, and by differentiation also.

The sequence $\langle 1, 0, 1, 0, \ldots \rangle$ is another one whose generating function can be obtained by many ways. We can obviously derive the formula

$$\sum_n z^{2n} = \frac{1}{(1-z)^2}$$

by substituting $z^2$ for $z$ in the identity

$$\sum_n z^n = \frac{1}{1-z} \;;$$

we can also apply cumulative summation to the sequence $\langle 1, -1, 1, -1, \ldots \rangle$, whose generating function is $\frac{1}{1+z}$, getting

$$\frac{1}{(1+z)(1-z)} = \frac{1}{(1-z^2)}.$$

# Simple sequences and their generating functions

| Sequence | Generating Function | Closed Form |
|---|---|---|
| $\langle 1, 0, 0, 0, \ldots \rangle$ | $\sum_{n\geq 0} [n=0]\ z^n$ | $1$ |
| $\langle 0, \ldots, 0, 1, 0, \ldots \rangle$ | $\sum_{n\geq 0} [n=m]\ z^n$ | $z^m$ |
| $\langle 1, 1, 1, 1, \ldots \rangle$ | $\sum_{n\geq 0} z^n$ | $\frac{1}{1-z}$ |
| $\langle 1, -1, 1, -1, \ldots \rangle$ | $\sum_{n\geq 0} (-1)^n z^n$ | $\frac{1}{1+z}$ |
| $\langle 1, 0, 1, 0, \ldots \rangle$ | $\sum_{n\geq 0} [2\backslash n] z^n$ | $\frac{1}{1-z^2}$ |
| $\langle 1, 0, \ldots, 0, 1, 0, \ldots, 0, 1, 0, \ldots \rangle$ | $\sum_{n\geq 0} [m\backslash n] z^n$ | $\frac{1}{1-z^m}$ |

# Simple sequences and their generating functions

| Sequence | Generating Function | Closed Form |
|---|---|---|
| $\langle 1, 2, 3, 4, 5, 6, \ldots \rangle$ | $\sum_{n \geq 0} (n+1) z^n$ | $\frac{1}{(1-z)^2}$ |
| $\langle 1, 2, 4, 8, 16, 32 \ldots \rangle$ | $\sum_{n \geq 0} 2^n z^n$ | $\frac{1}{1-2z}$ |
| $\langle 1, 4, 6, 4, 1, 0, 0, \ldots \rangle$ | $\sum_{n \geq 0} \binom{4}{n} z^n$ | $(1+z)^4$ |
| $\langle 1, c, \binom{c}{2}, \binom{c}{3} \ldots \rangle$ | $\sum_{n \geq 0} \binom{c}{n} z^n$ | $(1+z)^c$ |
| $\langle 1, c, \binom{c+1}{2}, \binom{c+2}{3} \ldots \rangle$ | $\sum_{n \geq 0} \binom{c+n-1}{n} z^n$ | $\frac{1}{(1-z)^c}$ |

# Simple sequences and their generating functions

| Sequence | Generating Function | Closed Form |
|---|---|---|
| $\langle 1, c, c^2, c^3, \ldots \rangle$ | $\sum_{n\geq 0} c^n z^n$ | $\frac{1}{1-cz}$ |
| $\langle 1, \binom{m+1}{m}, \binom{m+2}{m}, \binom{m+3}{m}, \ldots \rangle$ | $\sum_{n\geq 0} \binom{m+n}{m} z^n$ | $\frac{1}{(1-z)^{m+1}}$ |
| $\langle 0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \rangle$ | $\sum_{n\geq 1} \frac{1}{n} z^n$ | $\ln \frac{1}{1-z}$ |
| $\langle 0, 1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \ldots \rangle$ | $\sum_{n\geq 1} \frac{(-1)^{n+1}}{n} z^n$ | $\ln(1+z)$ |
| $\langle 1, 1, \frac{1}{2}, \frac{1}{6}, \frac{1}{24}, \frac{1}{120} \ldots \rangle$ | $\sum_{n\geq 1} \frac{1}{n!} z^n$ | $e^z$ |

# Solving Recurrences

Given a sequence $\langle g_n \rangle$ that satisfies a given recurrence, we seek a closed form for $g_n$ in terms of $n$. A solution of this problem via generating functions proceeds in four steps:

1. Write down a single equation that expresses $g_n$ in terms of other elements of the sequence. This equation should be valid for all integers $n$, assuming that $g_{-1} = g_{-2} = \cdots = 0$.
2. Multiply both sides of the equation by $z^n$ and sum over all $n$. This gives, on the left, the sum $\sum_n g_n z^n$, which is the generalization function $G(z)$. The right-hand side should be manipulated so that it becomes some other expression involving $G(z)$.
3. Solve the resulting equation, getting a closed form for $G(z)$.
4. Expand $G(z)$ into a power series and read off the coefficient of $z^n$; this is a closed form for $g_n$.

This method works because the sigle function $G(z)$ represents the entire sequence $\langle g_n \rangle$ in such a way that many manipulations are possible.

# Football victory problem

A group of $n$ fans of a "winning football team" throw their hats high into the air. The hats come back randomly, one hat to each of the $n$ fans.

**Question:** How many ways are there (denoted by $h(n, k)$) for exactly $k$ fans to get their own hats back?

For example, if $n = 4$ and if the hats and fans are named $A, B, C, D$, we denote

$$BCDA$$

when fans $A, B, C, D$ receive the hats $B, C, D, A$ respectively.

There are $4! = 24$ possible ways.

# Football victory problem

The number of rightful owners are as follows:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| ABCD | 4 | BACD | 2 | CABD | 1 | DABC | 0 |
| ABDC | 2 | BADC | 0 | CADB | 0 | DACB | 1 |
| ACBD | 2 | BCAD | 1 | CBAD | 2 | DBAC | 1 |
| ACDB | 1 | BCDA | 0 | CBDA | 1 | DBCA | 2 |
| ADBC | 1 | BDAC | 0 | CDAB | 0 | DCAB | 0 |
| ADCB | 2 | BDCA | 1 | CDBA | 0 | DCBA | 0 |

Therefore

$$\begin{aligned} h(4,4) &= 1 \\ h(4,3) &= 0 \\ h(4,2) &= 6 \\ h(4,1) &= 8 \\ h(4,0) &= 9. \end{aligned}$$

# Football victory problem

We can determine $h(n,k)$ by noticing that if the number of ways to choose $k$ lucky hat owners, namely $\binom{n}{k}$, times the number of ways to arrange the remaining $n-k$ hats so that none of them goes to the right owners, namely $h(n-k,0)$.

A permutation is called a **derangement** if it moves every item and the number of derangements of $n$ objects is sometimes denoted by symbol $n¡$ (read "$n$ subfactorial".)

Theorefore

$$h(n-k,0) = (n-k)¡,$$

and we have the general formula

$$h(n,k) = \binom{n}{k} h(n-k,0) = \binom{n}{k}(n-k)¡.$$

# Football victory problem

There is an easy way to get a recurrence, because the sum of $h(n,k)$ for all $k$ is the total number of permutations of $n$ hats;

$$n! = \sum_k h(n,k) = \sum_k \binom{n}{k} (n-k)¡. \qquad (3)$$

The problem can be solved with generating functions in an interesting way.

The equation (3) becomes

$$n! = \sum_k h(n,k) = \sum_k \frac{n!}{k!(n-k)!}(n-k)¡$$

hence

$$1 = \sum_{k=0}^{n} h(n,k) = \sum_{k=0}^{n} \frac{1}{k!} \, \frac{(n-k)¡}{(n-k)!}.$$

# Football victory problem

Let $D(z)$ be the generating function for the sequence $\langle \frac{n¡}{n!} \rangle$.

Since the sequence $\langle \frac{1}{n!} \rangle$ has $e^z$ as generarting function,

$$\frac{1}{1-z} = e^z \ D(z).$$

Solving for $D(z)$ gives

$$D(z) = \frac{1}{1-z} e^{-z} = \frac{1}{1-z} \Big( \frac{1}{0!} z^0 - \frac{1}{1!} z^1 + \frac{1}{2!} z^2 + \cdots \Big).$$

Equating coefficients of $z^n$ both sides, we get

$$\frac{n¡}{n!} = \sum_{k=0}^{n} \frac{(-1)^k}{k!}.$$

# Exercises 2.

2. *Find the generating function associated with the Fibonacci sequence $\langle F_n \rangle$ defined below and find $F_n$:*

$$\begin{aligned} F_0 &= F_1 = 1 \\ F_n &= F_{n-1} + F_{n-2} \quad \text{for } n \geq 2. \end{aligned}$$

3. *Solve the recurrence relation*

$$\begin{aligned} T_1 &= 1 \\ T_n &= 2T_{n-1} + 1 \quad \text{for } n \geq 2, \end{aligned}$$

*using generating function technique.*

4. *Solve the recurrence relation*

$$\begin{aligned} L_1 &= 2 \\ L_n &= L_{n-1} + n \quad \text{for } n \geq 2. \end{aligned}$$

# Odd and even-indexed terms

There is a method for extracting the even numbered terms $\langle g_0, 0, g_2, 0, g_4, 0, \ldots \rangle$ of any given sequence.

If we add $G(-z)$ and $G(z)$ we get

$$G(z) + G(-z) = \sum_n g_n(1 + (-1)^n)z^n = 2\sum_n g_n \; [n \text{ even }]z^n$$

therefore

$$\frac{G(z) + G(-z)}{2} = \sum_n g_{2n} z^{2n}.$$

The odd-numbered terms can be extracted in a similar way,

$$\frac{G(z) - G(-z)}{2} = \sum_n g_{2n+1} z^{2n+1}.$$

# Even-indexed Fibonacci numbers

We know that

$$\sum_n F_n z^n = \frac{z}{1 - z - z^2}.$$

Hence

$$\frac{z}{1 - 3z + z^2}$$

is the generating function of even-indexed Fibonacci numbers.

That is, the sequence $\langle F_0, F_2, F_4, F_6, \ldots \rangle = \langle 0, 1, 3, 8, \ldots \rangle$, has

$$\sum_n F_{2n} z^n = \frac{z}{1 - 3z + z^2}$$

as the generating function.

# References


1. **Graham, Knuth and Patashnik**, *"Concrete Mathematics – A Foundation for Computer Science"*, Pearson Education.
2. **Marko Petkovsek, Herbert S. Wilf and Doron Zeilberger**, "$A = B$", AK Peters Ltd., Wellesley, Massachusetts.
3. **Herbert S. Wilf**, *"Generatingfunctionology"*, Third Edition, AK Peters Ltd., Wellesley, Massachusetts.