Integer Functions

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Overview

Whole numbers constitute the backbone of discrete mathematics, and we often need to convert from fractions or arbitrary real numbers to integers.

Kenneth E. Iverson introduced the following notations, "floor" and "ceiling", in 1960s.

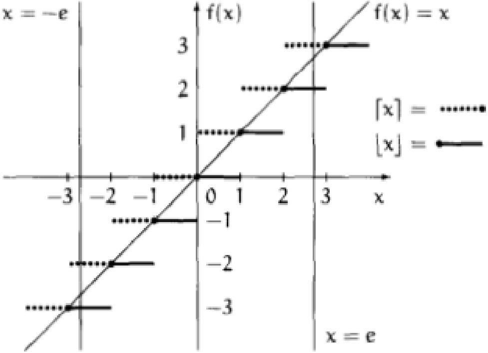
For any real x,

Floor Function $\lfloor x \rfloor =$ the greatest integer $\leq x$.

Ceiling Function $\lceil x \rceil =$ the least integer $\geq x$.

Floors and Ceilings

The graphs of floor and ceiling functions form staircase-like patterns above and below the digaonal line.



Floors and Ceilings

- Since the floor function lies on or below the diagonal line $f(x) = x$, we have $\lfloor x \rfloor \leq x$. Similarly, $\lceil x \rceil \geq x$.
- The two functions are equivalent precisely at the integer points.

$$\lfloor x \rfloor = x \iff x \text{ is an integer} \iff \lceil x \rceil = x.$$

- When they differ the ceiling is exactly 1 higher than the floor.

$$\lceil x \rceil - \lfloor x \rfloor = [x \text{ is not an integer}].$$

- $x - 1 < \lfloor x \rfloor \leq x \leq \lceil x \rceil < x + 1.$
- The functions are reflections of each other about both axes:

$$\lfloor -x \rfloor = -\lceil x \rceil; \quad \lceil -x \rceil = -\lfloor x \rfloor.$$

Thus each is easily expressible in terms of the other.

Floors and Ceilings

Mathematicians have long had both sine and cosine, tangent and cotangent, secant and cosecant, max and min; we discussed rising powers as well as falling powers. Now we have both floor and ceiling.

To prove properties about the floor and ceiling functions, the following **four rules** are especially useful.

Exercise

1. *Let n be an integer and x be real. Then prove that*
   - $\lfloor x \rfloor = n \iff n \leq x < n + 1$
   - $\lceil x \rceil = n \iff n - 1 < x \leq x$
   - $\lfloor x \rfloor = n \iff x - 1 < n \leq x$
   - $\lceil x \rceil = n \iff x \leq n < x + 1$
   - $\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1.$

Floor and ceiling brackets are comparatively inflexible

The first two are conseqences of definitions. While the rest are same with the first two inequalities, so that n is in the middle.

It is possible to move an integer term in or out of a floor (or ceiling) :

$$\lfloor x + n \rfloor = \lfloor x \rfloor + n,$$

for any integer n.

But moving out a constant factor cannot be done in general.

For example, $\lfloor nx \rfloor \neq n \lfloor x \rfloor$ when $n = 2$ and $x = 1/2$.

This means that floor and ceiling brackets are **comparatively inflexible**.

Inequality without floor or ceiling corresponds to same inequality with floor or with ceiling

But any inequality between a real and an integer is equivalent to a floor or ceiling inequality between integers :

$$\begin{aligned} x < n &\iff \lfloor x \rfloor < n \\ n < x &\iff n < \lceil x \rceil \\ x \leq n &\iff \lceil x \rceil \leq n \\ n \leq x &\iff n \leq \lfloor x \rfloor. \end{aligned}$$

Each inequality without floor or ceiling corresponds to same inequality with floor or with ceiling.

Integer and fractional parts of x

The difference between x and $\lfloor x \rfloor$ is called the **fractional part of** x denoted by $\{x\}$:

$$\{x\} = x - \lfloor x \rfloor.$$

We sometimes call $\lfloor x \rfloor$ the **integer part of** x, since $x = \lfloor x \rfloor + \{x\}$.

If a real number x can be written in the form

$$x = n + \theta$$

where n is an integer and $0 \leq \theta < 1$, then $n = \lfloor x \rfloor$ and $\theta = \{x\}$.

Exercises

2. *Prove that* $\lfloor x + n \rfloor = \lfloor x \rfloor + n$, *for any integer* n.
   Is $\lfloor x + n \rfloor = \lfloor x \rfloor + n$ *true, for an arbitrary real* n*?*
3. *Prove that for any real numbers* x *and* y, $\lfloor x + y \rfloor$ *is either* $\lfloor x \rfloor + \lfloor y \rfloor$ *or* $\lfloor x \rfloor + \lfloor y \rfloor + 1$. *In general,* $\lfloor x \rfloor + \lfloor y \rfloor \leq \lfloor x + y \rfloor \leq \lfloor x \rfloor + \lfloor y \rfloor + 1$.

Applications of Floor/Ceiling

Question

For a given positive integer n, how many bits are needed for $\lceil \lg n \rceil$ when we express the integer n in radix 2 notation?

We denote the base-2 logarithm of n by $\lg n$. Here powers of 2 play a vital role. If n satisfies $2^{m-1} \leq n < 2^m$ (n has m bits), then $m - 1 \leq \lg n < m$. Hence $m = \lfloor \lg n \rfloor + 1$.

That is, we need $\lfloor \lg n \rfloor + 1$ bits to express n in binary, for all $n > 0$.

Exercises

4. *Prove that $\lceil \lg(n+1) \rceil$ bits are needed to express n in binary, for all $n > 0$.*
5. *What is $\lceil \lfloor x \rfloor \rceil$ for any real number x?*
6. *Say true or false with justification : All expressions with an innermost $\lfloor x \rfloor$ surrounded by any number of floors or ceilings are same.*

Question

Prove or disprove the assertion

$$\lfloor \sqrt{\lfloor x \rfloor} \rfloor = \lfloor \sqrt{x} \rfloor, \text{ for any real } x \geq 0. \tag{1}$$

If x is an integer, then $x = \lfloor x \rfloor$. Hence (1) is true.

If nothing can be said about (1), we can think of giving counter example which shows that (1) is not true in general.

Even if the assertion (1) is true, our search for a counter example often leads us to a proof, as soon as we see why a counter example is impossible.

Question

Prove or disprove the assertion

$$\lfloor \sqrt{\lfloor x \rfloor} \rfloor = \lfloor \sqrt{x} \rfloor, \textit{ for any real } x \geq 0. \tag{2}$$

Suppose the assertion (2) is true. Then we have the following steps:

- start with $\lfloor \sqrt{\lfloor x \rfloor} \rfloor$
- strip off the outer floor, we get $\sqrt{\lfloor x \rfloor}$
- strip off the square root, we get $\lfloor x \rfloor$
- remove the inner floor, we get x
- take square root, we get $\sqrt{x}$
- add back the outer floor $\lfloor \sqrt{x} \rfloor$.

If we get the one what we started with, then (2) is proved.

Verification / Proof

$$
\begin{aligned}
m = \lfloor \sqrt{\lfloor x \rfloor} \rfloor \quad &\iff \quad m \leq \sqrt{\lfloor x \rfloor} < m + 1 \\
&\qquad \text{(since } \lfloor x \rfloor = n \iff n \leq x < n + 1\text{)} \\
&\iff \quad m^2 \leq \lfloor x \rfloor < (m + 1)^2 \\
&\qquad \text{since all the expressions are non-negative} \\
&\iff \quad m^2 \leq x < (m + 1)^2 \\
&\iff \quad m \leq \sqrt{x} < m + 1 \\
&\iff \quad m = \lfloor \sqrt{x} \rfloor
\end{aligned}
$$

Therefore $\lfloor \sqrt{\lfloor x \rfloor} \rfloor = \lfloor \sqrt{x} \rfloor$.

Exercise

7. *Prove that* $\lceil \sqrt{\lceil x \rceil} \rceil = \lceil \sqrt{x} \rceil$ *for any real* $x \geq 0$.

Theorem

Let $f(x)$ *be a any continuous monotonically increasing function with the property that if* $f(x)$ *is an integer, then* x *is an integer. Prove that*

$$\lfloor f(\lfloor x \rfloor) \rfloor = \lfloor f(x) \rfloor$$

and

$$\lceil f(\lceil x \rceil) \rceil = \lceil f(x) \rceil,$$

whenever $f(x), f(\lfloor x \rfloor)$ *and* $f(\lceil x \rceil)$ *are defined.*

Example

If $f(x) = \sqrt[k]{x}$, *then* $\lfloor \sqrt[k]{\lfloor x \rfloor} \rfloor = \lfloor \sqrt[k]{x} \rfloor$. *Verify* f *satisfies all assumptions in the above theroem for* f.

Proof of the theorem

We discuss the proof for the ceiling function.

If $x = \lceil x \rceil$, then there is nothing to prove.

Suppose $x < \lceil x \rceil$. Then $f(x) < f(\lceil x \rceil)$ since f is increasing. Then $\lceil f(x) \rceil < \lceil f(\lceil x \rceil) \rceil$ since $\lceil\ \rceil$ is non-decreasing.

Continuity of f gives that there is a **number** y such that $x \leq y < \lceil x \rceil$ and $f(y) = \lceil f(x) \rceil$. By a special propertiy of f (if $f(x)$ is an integer, then x is an integer), y is an **integer** such that $x \leq y < \lceil x \rceil$.

But there cannot be an integer strictly between x and $\lceil x \rceil$. This contradiction proves that $\lceil f(x) \rceil = \lceil f(\lceil x \rceil) \rceil$.

Exercises

8. *Prove that*

$$\lfloor \frac{x+m}{n} \rfloor = \lfloor \frac{\lfloor x \rfloor + m}{n} \rfloor$$

and

$$\lceil \frac{x+m}{n} \rceil = \lceil \frac{\lceil x \rceil + m}{n} \rceil$$

if m and n are integers and the denominator n is positive.

9. *Prove or disprove the statement*

$$\lceil \sqrt{\lfloor x \rfloor} \rceil = \lceil \sqrt{x} \rceil \quad \text{for any real } x \geq 0. \tag{3}$$

Does the assertion (3) work for $x = \pi$, $x = e$ *and* $x = \frac{1+\sqrt{5}}{2}$*?*

In Exercise (8), let $m = 0$; we have $\lfloor \lfloor \lfloor x/10 \rfloor \rfloor /10 \rfloor /10 = \lfloor x/1000 \rfloor$. Dividing thrice by 10 and throwing off digits is the same as dividing by 1000 and tossing the remainder

Level 0 : Something where no proof is required, only a lucky guess.

Level 1 : Given an **explicit object** X and an **explicit property** $p(x)$, prove that $p(x)$ is true.

For example, "Prove that $\lfloor \pi \rfloor = 3$." Proof involves arithmetic.

Level 2 : Given an **explicit set** X and an **explicit property** $p(x)$, prove that $p(x)$ is true for all $x \in X$.

For example, "Prove that $\lfloor x \rfloor \leq 3$ for all real x."

Proof must be general, should be algebraic, not just arithmetic.

Different levels of problems : Level 3

Level 3 : Given an **explicit set** X and an **explicit property** $p(x)$, prove or disprove that $p(x)$ is true for all $x \in X$.

For example, "Prove or disprove that $\lceil \sqrt{\lfloor x \rfloor} \rceil = \lceil \sqrt{x} \rceil$ for all real $x \geq 0$."

Here there is an additional level of uncertainty.

If the statement is false, our job is to find a counter example.

If the statement is true, we must find a proof as in level 2.

Different levels of problems : Level 4

Level 4 : Given an **explicit set** X and an **explicit property** $p(x)$, find a necessary and sufficient condition $q(x)$ that $p(x)$ is true.

For example, "Find a necessary and sufficient condition $\lfloor x \rfloor \geq \lceil x \rceil$."

The problem is to find an equivalent property (statement that is as simple as possible) for $p(x)$.

For example, in this case, "$\lfloor x \rfloor \geq \lceil x \rceil \iff x$ is an integer."

The extra element of discovery needed to find $q(x)$ makes this sort of problem more difficult.

Finally, a proof must be given that $p(x)$ is true iff $q(x)$ is true.

Level 5 : Given a **explicit set** X, find an **interesting property** $p(x)$ of its elements. **This is real mathematics.**

Whereas the statement $\lceil \sqrt{\lfloor x \rfloor} \rceil = \lceil \sqrt{x} \rceil$ for any real $x \geq 0$ can be converted from level 3 to level 4, as follows :

A necessary and sufficient condition that $\lceil \sqrt{\lfloor x \rfloor} \rceil = \lceil \sqrt{x} \rceil$ is either x is an integer or $\sqrt{\lfloor x \rfloor}$ is not.

Various Types of Intervals

The set of real numbers x such that $\alpha \leq x \leq \beta$, denoted by $[\alpha, \beta]$, is called a **closed interval** because it contains both endpoints α and β.

The set of real numbers x such that $\alpha < x < \beta$, denoted by (α, β), is called an **open interval**.

And intervals $[\alpha, \beta)$ and $(\alpha, \beta]$ which contain just one endpoint are defined similarly, are called **half-open**.

Half-open intervals are almost always nicer than open or closed intervals. For example, **they are additive** – we can combine the half-open intervals $[\alpha, \beta)$ and $[\beta, \gamma)$ to form the half-open interval $[\alpha, \gamma)$.

This would not work with open intervals because the point β would be excluded, and it could cause problems with closed intervals because β would be included twice.

Question

How many integers are contained in half-open intervals?

If α and β are integers, then $[\alpha, \beta)$ contains the $\beta - \alpha$ integers $\alpha, \alpha + 1, \ldots, \beta - 1$, assuming that $\alpha \leq \beta$. Similarly, (α, β) contains $\beta - \alpha$ integers in such a case.

Suppose α and β are arbitrary reals. We have

$$\begin{aligned} \alpha \leq n < \beta \quad &\iff \quad \lceil \alpha \rceil \leq n < \lceil \beta \rceil \\ \alpha < n \leq \beta \quad &\iff \quad \lfloor \alpha \rfloor < n \leq \lfloor \beta \rfloor, \end{aligned}$$

where n is an integer. The intervals $[\alpha, \beta)$ contains exactly $\lceil \beta \rceil - \lceil \alpha \rceil$ integers, and $(\alpha, \beta]$ contains $\lfloor \beta \rfloor - \lfloor \alpha \rfloor$ because

- the intervals on the right have integer end points,
- the intervals on the left have real end points, and
- both intervals are having the same number of integers.

By the way, there is a **mnemonic** for remembering which case uses floors and which uses ceilings.

So by Murphys's law, the correct result is the opposite of what we would expect – ceilings for $[\alpha, \beta)$ and floors for $(\alpha, \beta]$.

Exercise

10. *Prove the following:*
    - $[\alpha, \beta]$ *(*$\alpha \leq \beta$*) contains* $\lfloor \beta \rfloor - \lceil \alpha \rceil + 1$ *integers,*
    - $(\alpha, \beta]$ *(*$\alpha \leq \beta$*) contains exactly* $\lfloor \beta \rfloor - \lfloor \alpha \rfloor$ *integers,*
    - (α, β) *(*$\alpha < \beta$*) contains exactly* $\lceil \beta \rceil - \lfloor \alpha \rfloor - 1$ *integers.*

There is a casino in which there is a roulette wheel with one thousand slots, numbered 1 to 1000.

If the number n that comes up on a spin is divisible by the floor of its cube root (n is a multiple of $\lfloor\sqrt[3]{n}\rfloor$), then **the number is a winner**, we get Rs.5, otherwise, we lose Re.1.

We can compute the average winnings. If W is the number of winners during 1000 plays, then $L = 1000 - W$ is the number of losers.

If each number **comes up once** during 1000 plays, we win Rs.$5W$ and lose Rs.L, so the average winning will be

$$\frac{5W - L}{1000} = \frac{5W - (1000 - W)}{1000} = \frac{6W - 1000}{1000}.$$

Question

How can we count the number of winners among 1 *through* 1000*?*

The numbers from 1 through $2^3 - 1 = 7$ are all winners because $\lfloor \sqrt[3]{n} \rfloor = 1$ for each.

Among the numbers $2^3 = 8$ through $3^3 - 1 = 26$, only the even numbers are winners.

And among $3^3 = 27$ through $4^3 - 1 = 63$, only those divisible by 3 are. And so on.

By Iverson's convention, we have

$$
\begin{aligned}
W &= \sum_{n=1}^{1000}[n \text{ is a winner}] = \sum_{1\le n\le 1000} [\lfloor \sqrt[3]{n} \rfloor \text{ divides } n] \\
&= \sum_{k,n}[k = \lfloor \sqrt[3]{n} \rfloor]\ [k \text{ divides } n]\ [1 \le n \le 1000] \\
&= \sum_{k,n,m}[k^3 \le n < (k+1)^3]\ [n = km]\ [1 \le n \le 1000] \\
&= 1 + \sum_{k,m}[k^3 \le km < (k+1)^3]\ [1 \le k < 10]\ \left\{\text{Note that } m = 100 \text{ when } k = 10\right\} \\
&= 1 + \sum_{k,m}[m \in [k^2, \frac{(k+1)^3}{k})]\ [1 \le k < 10] \\
&= 1 + \sum_{1\le k<10} \left(\lceil \frac{(k+1)^3}{k} \rceil - \lceil k^2 \rceil\right) \\
&= 1 + \sum_{1\le k<10} \left(\lceil k^2 + 3k + 3 + 1/4 \rceil - \lceil k^2 \rceil\right) \\
&= 1 + \sum_{1\le k<10} (3k+4) = 1 + \frac{7+31}{2} 9 = 172.
\end{aligned}
$$

We have answered the following question :

Question

How many integers n, where $1 \le n \le 1000$, *satisfy the relation* $\lfloor \sqrt[3]{n} \rfloor / n$?

Let us discuss the general case for large number N. We denote $R = \lfloor \sqrt[3]{N} \rfloor$.

The total number of winners for general N comes to

$$\begin{aligned} W &= \sum_{1 \le k < R} (3k+4) + \sum_m [R^3 \le Rm \le N] \\ &= \frac{1}{2}(7 + 3R + 1)(R-1) + \sum_m \left[[m \in [R^2, N/R] \right] \\ &= \frac{3}{2}R^2 + \frac{5}{2}R - 4 + \sum_m \left[[m \in [R^2, N/R] \right] \\ &= \frac{3}{2}R^2 + \frac{5}{2}R - 4 + \lfloor N/R \rfloor - \lceil R^2 \rceil + 1 \\ &= \frac{3}{2}R^2 + \frac{5}{2}R - 4 + \lfloor N/R \rfloor - R^2 + 1. \end{aligned}$$

Hence the formula (for number of winners) is

$$W = \lfloor N/R \rfloor + \frac{1}{2}R^2 + \frac{5}{2}R - 3$$

where $R = \lfloor \sqrt[3]{N} \rfloor$, gives the general answer for a **wheel of size N.**

We define the **spectrum of a real number** α to be an infinite multiset (a set can have repeated elements) of integers :
Spec(α)=$\{\lfloor\alpha\rfloor, \lfloor 2\alpha\rfloor, \lfloor 3\alpha\rfloor, \ldots\}$.

Exercise

11. *Find Spec($\frac{1}{2}$), Spec($\sqrt{2}$) and Spec($2+\sqrt{2}$).*

Theorem

Prove that Spec(α) $\neq$ Spec(β), when $\alpha \neq \beta$. That is, distinct real numbers have distinct spectra.

Proof. Without loss of generality, we assume that $\alpha < \beta$. Then there exists a positive integer m such that $m(\beta - \alpha) \geq 1$. Hence $m\beta - m\alpha \geq 1$ and $\lfloor m\beta\rfloor > \lfloor m\alpha\rfloor$. Spec($\beta$) has fever than m elements $\leq \lfloor m\alpha\rfloor$ while Spect(α) has at least m.

Example

- *Spec(1)*=$\{1, 2, 3, \ldots\}$
- *Spec(3)*=$\{3, 6, 9, \ldots\}$
- *Spec($\frac{1}{2}$)*=$\{0, 1, 1, 2, 2, 3, 3, 4, \ldots\}$
- *Spec($\frac{1}{k}$)*=$\Big\{0, \ldots, 0\ (k-1\ \textit{times}), 1, \ldots, 1\ (k\ \textit{times}),\Big\}$

- Suppose $\lfloor m\alpha \rfloor = k$. Then $k \le m\alpha < k+1$ which implies that $\frac{k}{m} \le \alpha < \frac{k}{m} + \frac{1}{m}$.
- If $\alpha \ge 1$, then spec(α) is an ordinary set (not a multiset), **no repetitions**.
- If $m \ne n$, then $\lfloor m\alpha \rfloor \ne \lfloor n\alpha \rfloor$.
- Let α be real > 1. Then the fractional parts of $m\alpha$ are all distinct $(m \ge 1)$ iff α is irrational.
- If α is irrational, them the fractional parts of $m\alpha$ are uniformly distributed in $(0, 1)$.

Partition of the positive integers

If the set of positive integers is the disjoint union of Spec(α) and Spec(β), then we say that these spectra form a **partition** of the positive integers.

We denote the number of elements in Spec(α) that are $\leq n$ by $N(\alpha, n)$.

If we show that

$\boxed{\text{Spec}(\alpha) \text{ and } \text{Spec}(\beta) \text{ are disjoint, and } N(\alpha, n) + N(\beta, n) = n,}$ then Spec(α) and Spec(β) form a partition of the positive integers.

We can find $N(\alpha, n)$ as follows :

$$\begin{aligned} N(\alpha, n) &= \sum_{k>0} [\lfloor k\alpha \rfloor \leq n] \\ &= \sum_{k>0} [\lfloor k\alpha \rfloor < n+1] \end{aligned}$$

since $m \leq n \iff m < n+1$ for all integers m, n.

This law is used to change "$\leq$" to "$<$", so that floor bracket can be removed in the next step.

$$
\begin{aligned}
N(\alpha, n) &= \sum_{k>0} [k\alpha < n+1] \\
&= \sum_{k>0} [0 < k < (n+1)/\alpha] \\
&= \lceil (n+1)/\alpha \rceil - 1
\end{aligned}
$$

Question

Do Spec($\sqrt{2}$) and Spec($2+\sqrt{2}$) form a partition of the positive integers?

Verification: Let n be a positive integer.

$$
\begin{aligned}
N(\sqrt{2}, n) + N(2+\sqrt{2}, n) = n \quad & \Longleftrightarrow \quad \left\lceil \frac{n+1}{\sqrt{2}} \right\rceil - 1 + \left\lceil \frac{n+1}{2+\sqrt{2}} \right\rceil - 1 = n \\
& \Longleftrightarrow \quad \left\lfloor \frac{n+1}{\sqrt{2}} \right\rfloor + \left\lfloor \frac{n+1}{2+\sqrt{2}} \right\rfloor = n \\
& \Longleftrightarrow \quad \frac{n+1}{\sqrt{2}} - \left\{ \frac{n+1}{\sqrt{2}} \right\} + \frac{n+1}{2+\sqrt{2}} - \left\{ \frac{n+1}{2+\sqrt{2}} \right\} = n \\
& \Longleftrightarrow \quad \left\{ \frac{n+1}{\sqrt{2}} \right\} + \left\{ \frac{n+1}{2+\sqrt{2}} \right\} = 1
\end{aligned}
$$

$\Longleftrightarrow$ The above assertion is true because these are the fractional parts of 2 non-integers that add up to the integer $n+1$.

Thus Spec($\sqrt{2}$) and Spec($2+\sqrt{2}$) form a partition of the positive integers.

Floor/Ceiling Recurrence

Consider the recurrence relation

$$\begin{aligned} K_0 &= 1 \\ K_{n+1} &= 1 + \min\{2K_{\lfloor n/2 \rfloor}, 3K_{\lfloor n/3 \rfloor}\} \quad \text{for } n \geq 0. \end{aligned}$$

The sequence of numbers $1, 3, 3, 4, 7, 7, 7, 9, 9, 10, 13, \ldots$ are called the **knuth** numbers.

We discuss the following the statement:

Question

Prove or disprove that $K_n \geq n$, for all $n \geq 0$.

The first few K's just listed do satisfy the inequality, so there is a good chance that it is true in general.

Let us try an induction proof.

$K_0 = 1 \geq 0$.

Assume that the inequality holds for all values up through some fixed non-negative integer n.

Claim : $K_{n+1} \geq n + 1$.

$K_{n+1} = 1 + \min\{2K_{\lfloor n/2 \rfloor}, 3K_{\lfloor n/3 \rfloor}\}$ since $n/2 < n$, $K_{\lfloor n/2 \rfloor} \geq \lfloor n/2 \rfloor$ and $K_{\lfloor n/3 \rfloor} \geq 3\lfloor n/3 \rfloor$.

However, $2\lfloor n/2 \rfloor$ can be as small as $n - 1$, and $3\lfloor n/3 \rfloor$ can be as small as $n - 2$.

$$\begin{aligned} K_{n+1} &= 1 + \min\{2K_{\lfloor n/2 \rfloor}, 3K_{\lfloor n/3 \rfloor}\} \\ &\geq 1 + \min\{n - 1, n - 2\} \\ &= 1 + n - 2 = n - 1. \end{aligned}$$

This falls far short for $K_{n+1} \geq n + 1$.

Let us now try to disprove it. If we can find an integer n_0 such that either $2\lfloor n_0/2 \rfloor < n_0$ or $3\lfloor n_0/3 \rfloor < n_0$, then we get

$$K_{n_0+1} = 1 + \min\{2K_{\lfloor n_0/2 \rfloor}, 3K_{\lfloor n_0/3 \rfloor}\} < 1 + n_0.$$

Is this possible to find such an integer n_0?

Question

Why recurrence relation (involving floors and/or ceilings) is needed?

Recurrence relations involving floors and/or ceilings arise often in computer science, because algorithms based on the important technique of "**divide and conquer**" often reduce a problem of size n to the solution of similar problems of integer sizes that are fraction of n.

For example, one way to sort n records if $n > 1$, is to divide them into two approximately equal parts, one of size $\lceil n/2 \rceil$ and the other of size $\lfloor n/2 \rfloor$. Note that $\lceil n/2 \rceil + \lfloor n/2 \rfloor = n$.

After each part has been sorted separately (by the same method, applied recursively), we can merge the records into their final order by doing at most $n - 1$ further comparisons.

Therefore the total number of comparisons performed is at most $f(n)$, where

$$\begin{aligned} f(1) &= \; = 0 \\ f(n) &= f(\lceil n/2 \rceil) + f(\lfloor n/2 \rfloor) + n - 1 \quad \text{for } n > 1. \end{aligned}$$

The Josephus problem has a similar recurrence, which can be cast in the form

$$\begin{aligned} J(1) &= 1 \\ J(n) &= 2J(\lfloor n/2 \rfloor) - (-1)^n \quad \text{for } n > 1. \end{aligned}$$

Let us consider the actual Josephus problem in which every third person is eliminated, instead of every second.

We have a recurrence

$$J_3(n) = \lceil \frac{3}{2} J_3\Big(\lfloor \frac{2}{3} n \rfloor\Big) + a_n \rceil \bmod n + 1,$$

where $a_n = -2, +1$, or $-\frac{1}{2}$ according as $n \bmod 3 = 0, 1$, or 2. But this recurrence **is too horrible** to purse.

Another approach to the Josephus problem

There is another approach to the Josephus problem that gives a much better setup.

Whenever a person is passed over, we can assign a new number.

Thus, 1 and 2 become $n + 1$ and $n + 2$, then 3 is executed;

4 and 5 become $n + 3$ and $n + 4$, then 6 is executed;

$\vdots$

$3k + 1$ and $3k + 2$ become $n + 2k + 1$ and $n + 2k + 2$, then $3k + 3$ is executed;

$\vdots$

then $3n$ is executed (or left to survive).

Another approach to the Josephus problem

For example, when $n = 10$ the numbers are

1	2	3	4	5	6	7	8	9	10
11	12		13	14		15	16		17
18			19	20			21		22
			23	24					25
			26						27
			28						
			29						
			30						

The kth person eliminated ends up with number $3k$. So we can figure out who survivor is if we can figure out the original number of person numbered $3n$.

If $N > n$, person numbered N must have had a pervious number, and we can find it as follows:

We have $N = n + 2k + 1$ or $N = n + 2k + 2$, hence $k = \lfloor (N - n - 1)/2 \rfloor$; the pervious number was $3k + 1$ or $3k + 2$, respectively.

That is, it was $3k + (N - n - 2k) = k + N - n$. Hence we can calculate the survivor's number $J_3(n)$ as follows:

$N := 3n$;

while $N > n$ do $N := \lfloor \frac{N-n-1}{2} \rfloor + N - n$;

$J_3(n) := N$.

This is not a closed form for $J_3(n)$; it is not even a recurrence. But at least it tells us how to calculate the answer reasonably fast, if n is large.

Solution to the Josephus problem

Fortunately there is a way to simplify this algorithm if we use the variable $D = 3n + 1 - N$ in place of N. This change in notation corresponds to assigning numbers from $3n$ down to 1, instead of from 1 up to $3n$.

Then the complicated assignment to N becomes

$$\begin{aligned} D &:= 3n + 1 - \left(\lfloor \frac{(3n+1-D) - n - 1}{2} \rfloor + (3n + 1 - D) - n \right) \\ &= n + D - \lfloor \frac{2n - D}{2} \rfloor = D - \lfloor \frac{-D}{2} \rfloor = D + \lceil \frac{D}{2} \rceil = \lceil \frac{3}{2} D \rceil, \end{aligned}$$

and we can rewrite the algorithm as follows:

$D := 1;$

while $D \leq 2n$ do $D := \lceil \frac{3}{2} D \rceil;$

$J_3(n) := 3n + 1 - D.$

Solution to the generalized Josephus problem

This looks much nicer, because n enters the calculation in a very simple way.

In fact, we can show by the same reasoning that the survivor $J_q(n)$ when every qth person is eliminated can be calculated as follows:

$$D := 1;$$

$$\textbf{while } D \leq (q-1)n \ \text{ do } D := \left\lceil \tfrac{q}{q-1} D \right\rceil;$$

$$J_q(n) := qn + 1 - D.$$

When $q = 2$, this makes D grow to 2^{m+1} when $n = 2^m + \ell$.

Hence $J_2(n) = 2(2^m + \ell) + 1 - 2^{m+1} = 2\ell + 1$.

Solution to the generalized Josephus problem

We compute a sequence of integers that can be defined by the following recurrence

$$\begin{aligned} D_0^{(q)} &= 1 \\ D_n^{(q)} &= \left\lceil \frac{q}{q-1} D_{n-1}^{(q)} \right\rceil \quad \text{for } n > 0. \end{aligned}$$

These numbers $D_n^{(q)}$ for $n \geq 0$ do not seem to relate to any familiar functions in a simple way, expect when $q = 2$; hence they probably do not have a nice closed form.

But if we are willing to accept the sequence $D_n^{(q)}$ as "known," then it is easy to describe the solution to the generalized Josephus problem:

The survivor $J_q(n)$ is $qn + 1 - D_k^{(q)}$, where k is as small as possible such that $D_k^{(q)} > (q-1)n$.

Exercises

12. *In Josephus problem, we represented an arbitrary positive number n in the form $n = 2^m + \ell$, where $0 \leq \ell < 2^m$. Give explicit formulas for ℓ and m as functions of n, using floor and / or ceiling brackets.*
13. *What is a formula for the nearest integer to a given real number x? In case of ties, when x is exactly halfway that rounds*
    (a) *up, that is, to $\lceil x \rceil$*
    (b) *down, that is, to $\lfloor x \rfloor$.*
14. *Evaluate $\lfloor \lfloor m\alpha \rfloor n/\alpha \rfloor$, when m and n are positive integers and α is an irrational number greater than n.*
15. *Find a necessary and sufficient condition that For example, $\lfloor nx \rfloor \neq n\lfloor x \rfloor$ when n is a positive integer.*
16. *Prove the* Dirichlet box principle *:*
    If n objects are put into m boxes, some box must contain $\geq \lceil n/m \rceil$ objects, and some box must contain $\leq \lfloor n/m \rfloor$.

Exercises

17. *Can something be said about $\lfloor f(x) \rfloor$ when $f(x)$ is a continuous monotonically decreasing function that takes integer values only when x is an integer?*
18. *Show that the expression*

$$\left\lceil \frac{2x+1}{2} \right\rceil - \left\lceil \frac{2x+1}{4} \right\rceil + \left\lfloor \frac{2x+1}{4} \right\rfloor$$

is always either $\lfloor x \rfloor$ or $\lceil x \rceil$. In what circumstances does each case arise?

19. *Let α and β be positive real numbers. Prove that Spec(α) and Spec(β) partition the positive integers if and only if α and β are irrational and $\frac{1}{\alpha} + \frac{1}{\beta} = 1$.*
20. *Find a necessary and sufficient condition on the real number $b > 1$ such that $\lfloor \log_b x \rfloor = \lfloor \log_b \lfloor x \rfloor \rfloor$ for all real $x \geq 1$.*

Exercises

21. *Find the sum of all multiples of x in the closed interval $[\alpha, \beta]$, when $x > 0$.*
22. *How many of the numbers 2^m, for $0 \leq m \leq M$, have leading digit 1 in decimal notation?*
23. *Evaluate the sums*
$$S_n = \sum_{k \geq 1} \lfloor \frac{n}{2^k} + \frac{1}{2} \rfloor$$
and
$$T_n = \sum_{k \geq k} 2^k \lfloor \frac{n}{2^k} + \frac{1}{2} \rfloor^2.$$
24. *Show that $\lfloor \sqrt{2n} + \frac{1}{2} \rfloor$ is the nth element of the sequence*
$$1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 5, 5, 6 \cdots .$$
The sequence has exactly k occurrences of 'k', for $k > 1$.

Exercises

25. *Prove that for any real x and positive integer m,*

$$\lfloor \frac{\lfloor x \rfloor}{m} \rfloor = \lfloor \frac{x}{m} \rfloor.$$

26. *Prove that for any real x,*
    (a) $\lfloor x \rfloor + \lfloor x + \frac{1}{2} \rfloor = \lfloor 2x \rfloor$
    (b) $\lfloor x + \frac{1}{3} \rfloor + \lfloor x + \frac{2}{3} \rfloor = \lfloor 3x \rfloor$
    (c) $\lfloor x \rfloor + \lfloor x + \frac{1}{m} \rfloor + \lfloor x + \frac{2}{m} \rfloor + \cdots + \lfloor x + \frac{m-1}{m} \rfloor = \lfloor mx \rfloor.$
27. *Prove that spectrum of $\sqrt{2}$ contains infinitely many powers of 2. That is, prove that there are infinitely many integers $n \geq 1$ such that $\lfloor n\sqrt{2} \rfloor = 2^k$ for some $k > 0$.*

Exercises

28. *Solve the recurrence*

$$X_n = n, \quad \text{for } 0 \le n < m,$$
$$X_n = X_{n-m} + 1, \quad \text{for } n \ge m.$$

29. *Solve the recurrence*

$$a_0 = 1,$$
$$a_n = a_{n-1} + \lfloor \sqrt{a_{n-1}} \rfloor, \quad \text{for } n > 0.$$

30. *Let α and β be positive real numbers. We have proved that Spec(α) and Spec(β) partition the positive integers if and only if α and β are irrational and $\frac{1}{\alpha} + \frac{1}{\beta} = 1$.*
This establishes an interesting relation *between the two multisets Spec(α) and Spec($\alpha/(\alpha-1)$), when α is any irrational > 1, because and $\frac{1}{\alpha} + \frac{\alpha-1}{\alpha} = 1$.*
Find (and prove) an interesting relation *between the two multisets Spec(α) and Spec($\alpha/(\alpha+1)$), where α is any positive real number.*

Exercises

31. *Prove or disprove :* $\lfloor x\rfloor + \lfloor y\rfloor + \lfloor x+y\rfloor \le \lfloor 2x\rfloor + \lfloor 2y\rfloor$.
32. *Prove that*
$$\sum_{k=1}^{\infty}\lfloor \frac{n}{2^k} + \frac{1}{2}\rfloor = n.$$
33. *Let* $\|x\| = \min(x-\lfloor x\rfloor, \lceil x\rceil - x)$ *denote the distance from* x *to the nearest integer.*
What is the value of
$$\sum_k 2^k \|x/2^k\|^2?$$

References

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