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Concrete Mathematics - MA 201
Problem Sheet - 7

1. What is the smallest positive integer that has exactly k divisors, for $1 \leq k \leq 6$?

2. Prove that $gcd(m,n).lcm(m,n) = m.n$, and use this identity to express $lcm(m,n)$ in terms of $lcm(n\ mod\ m, m)$, when $n\ modm \neq 0$.

3. Let $\pi(x)$ be the number of primes not exceeding x. Prove that

$$\pi(x) - \pi(x-1) = [\lfloor x \rfloor \text{ is prime}].$$

4. Ten people numbered 1 to 10 are lined up in a circle as in the Josephus problem, and every mth person is executed. (The value of m may be much larger than 10.) Prove that the first three people to go cannot be $10, k$, and $k+1$ (in this order), for any k.

5. The residue number system $(x\ mod\ 3, x\ mod\ 5)$ has the curious property that 13 corresponds to $(1,3)$, which looks almost the same. Explain how to find all instances of such a coincidence, withoug calculating all fifteen pairs of residues. In other words, find all solutions of the congruences

$$10x + y \equiv x\ (mod\ 3), \qquad 10x + y \equiv y\ (mod\ 5).$$

6. Show that $(3^{77} - 1/2$ is odd and composite.

7. Compute $\phi(999)$.

8. A positive integer n is called **squarefree** if it is not divisible by m^2, for any $m > 1$. Find a necessary and sufficient condition that n is squarefree, in terms of the prime-exponent representation of n.

9. Prove that when $k > 0$
   - $gcd(km, kn) = k\,gcd(m,n)$;
   - $lcm(km, kn) = k\,lcm(m,n)$.

10. Let f_n be the "Fermat number" $2^{2^n} + 1$. Prove that $gcd(f_m, f_n) = 1$ if $m < n$.

11. Show that if $2^n + 1$ is prime then n is a power of 2.

12. The number 111 111 111 111 111 1111 is prime. Prove that, in any radix b, $(11\ldots1)_b$ can be prime only if the number of 1's is prime.

13. State a recurrence for $\rho(k)$, the ruler function function. Show that there is a connection between $\rho(k)$ and the dist that is moved at step k when an n-disk "Tower of Hanoi" is being transferred in $2^n - 1$ moves, for $1 \leq k \leq 2^n - 1$.

14. Express $\varepsilon_p(n!)$ in terms of $v_p(n)$, the sum of the digits in the radix p representation of n.

15. We say that m **exactly divides** n, written $m\backslash\backslash n$, if $m\backslash n$ and $gcd(m, n/m) = 1$. For example, $p^{\varepsilon_p(n!)}\backslash\backslash n!$. Prove or disprove the following:

    - $k\backslash\backslash n$ and $m\backslash\backslash n \iff km\backslash\backslash n$, if $gcd(k, m) = 1$.
    - For all $m, n > 0$, either $gcd(m, n)\backslash\backslash m$ or $gcd(m, n)\backslash\backslash n$.

16. A number in decimal notation is divisible by 3 if and only if the sum of its digits is divisible by 3. Prove this well-known rule, and generalize it.

17. Prove that if $gcd(a, b) = 1$ and $a > b$, then

$$gcd(a^m - b^m, a^n - b^n) = a^{gcd(m,n)} - b^{gcd(m,n)}, \qquad 0 \leq m < n.$$

    All variables are integers.

18. Show that if $p \; mod \; 4 = 3$, there is no integer n such that p divides $n^2 + 1$. But show that if $p \; mod \; 4 = 1$, there is such an integer.

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