Matrices and Gaussian Elimination : Part 2

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Example of Gaussian Elimination

Given a system $Ax = \begin{pmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \\ 9 \end{pmatrix} = b.$

First step: subtract 2 times the first equation from the **second**. The **elementary matirx** $E = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ should be pre-multiplied in $Ax = b$, we get

$$EAx = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 5 \\ -2 \\ 9 \end{pmatrix}.$$

Our original matrix subtracts 2 times the first component from the second, leaving the first and third components unchanged. After this step the new and simple system (equivalent to the old) is just $E(Ax) = Eb$.

The matrix that leaves every vector unchanged is the identity matrix I, with 1's on the diagonal and 0's everywhere else. The matrix that subtracts a multiple ℓ of row j from row i is the elementary matrix E_{ij}, with 1's on the diagonal and $-\ell$ in row i, column j.

For instance, $E_{31} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\ell & 0 & 1 \end{pmatrix}$, then $E_{31} b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 - \ell b_1 \end{pmatrix}$.

In the above example, there are three elimination steps:

1. subtract 2 times the first equation from the **second**
2. subtract -1 times the first equation from the **third**
3. subtract -1 times the second equation from the **third**.

The result is an equivalent but simpler system, with a new coefficient matrix U (**upper triangular matrix**)

$$Ux = \begin{pmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 5 \\ -12 \\ 2 \end{pmatrix} = c.$$

The elementary matrices for steps (i), (ii) and (iii) repectively are

$$E = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, F = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} G = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

The result of all three steps $GFEA = U$, where

$$GFE = L = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix}$$ is a **lower triangular matrix**.

We could multiply GFE together to find the single matrix that takes A to U (and also takes b to c).

Thus $LA = U$.

This is good, but the most important question is exactly the opposite. How would we get from U back to A? How can we undo the steps of Gaussian elimination.

A single step, say step (a), is not hard to undo. Instead of subtracting, we add twice the first row to the second. (Not twice the second row to the first!) The result of doing both the subtraction and the addition is to bring back the identity matrix.

$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. If the **elementary matrix** E has the number $-\ell$ in the (i, j) position, then its inverse has $+\ell$ in that position. That matrix is denoted by E^{-1}. Thus E^{-1} times E is the identity matrix.

The final problem is to undo the whose process at once, and the matrix $E^{-1}F^{-1}G^{-1}$ takes U back to A ($E^{-1}F^{-1}G^{-1}U = A$). It is the link between the A we start with and the U we reach. It is called L, (for instance, $L = E^{-1}F^{-1}G^{-1}$) because it is lower triangular. L has special property: the entries below the diagonal are exactly the multipliers, $\ell = 2, -1$ and -1. Thus $A = L^{-1}U$.

Triangular factorization $A = LU$. If no row exchanges are required, the orignial matrix A can be written as a product $A = LU$. The matrix L is lower triangular, with 1's on the diagonal and the multipliers ℓ_{ij} (taken from elimination) below the diagonal. U is the upper triangular matrix which appears after forward elimination and before back-substitution; its diagonal entries are **the pivots**.

The rule is that the matrix L, applied to U, brings back A.

One linear system = two triangular system

When L and U are known, A could be thrown away. We go from b to c by forward elimination (that uses L) and we go from c to x by back-substitution (that uses U). We can and should do without A, when its factors have been found. ($A = Lu$, $b = Ax = LUx$ implies $L^{-1}Ax = L^{-1}b = c$).

In matrix terms, elimination splits $Ax = b$ into two triangular systems: first $Lc = b$ and then $Ux = c$. This identical to $Ax = b$. Pre-multiply $Ux = c$ by L to give $LUx = Lc$, which is $Ax = b$. Each triangular system can be solved in $n^2/2$ steps. The solution for any new right side b' can be found in only n^2 operations. That is far below the $n^3/3$ steps needed to factor A on the left hand side.

The triangular factorization is often written $A = LDU$, where L and U have 1's on the diagonal and D is the diagonal matrix of pivots.

It is conventional, although completely confusing, to go on denoting this new upper triangular matrix by the same letter U. Whenever you see LDU, it is understood that U has 1's on the diagonal - in other words that each row was divided by the pivot. Then L and U are treated evenly. An example for $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ is

$$A = \begin{pmatrix} 1 & \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ & -2 \end{pmatrix} = \begin{pmatrix} 1 & \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & \\ & -2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ & 1 \end{pmatrix} = LDU.$$

That has the 1's on the diagonals of L and U, and the pivots 1 and -2 in D.

Unique triangular factorization

If $A = L_1 D_1 U_1$ and $A = L_2 D_2 U_2$, where the L's are lower triangular with unit diagonal, the U's are upper triangular with unit diagonal, and the D's are diagonal matrices with no zeros on the diagonal, then

$$L_1 = L_2, D_1 = D_2, U_1 = U_2.$$

The LDU factorization and the LU factorization are uniquely determined by A.

Row exchanges and permutation matrices.

We have to face a problem that the number we expect to use as a pivot might be zero. This could occur in the middle of a calculation, or it can happen at the very beginning (in case $a_{11} = 0$.)

A simple example is $\begin{pmatrix} 0 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$. The difficulty is clear, no multiple of the first equation will remove the coefficient 3.

The remedy is equally clear. **Exchange the two equations**, moving the entry 3 up into the pivot. To express this in matrix terms, we need to find **the permutation matrix that produces the row exchange**. It is $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and multiply by P does exchange the rows.

The next difficult case is that **a zero in the pivot location** raises two possibilites: the trouble may be easy to fix, or it may be serious.

This is decided by looking below the zero. If there is a nonzero entry lower down in the same column, then a row exchange is carried out; the nonzero entry becomes the needed pivot, and estimation can get going again.

In the example $A = \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ d & e & f \end{pmatrix}$, everything depends on the number d.

If $d = 0$, the problem is incurable and matrix is **singular**.

There is no hope for a unique solution. If d is not zero, an exchange of rows 1 and 3, permutation matrix $P_{13} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, will move d into the pivot, and stage 1 is complete. However the next pivot position also contains a zero.

The number a is now below it (the e above is useless) and if a is not zero, then another row exchange is callef for. $P_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ (exchange of rows 2 and 3).

There is a permutation matrix that will do both of the row exchanges at once, which is the product of the two separate permutaions,

$P_{23}P_{13} = P = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ (first exchange rows 1 and 3, then exchange rows 2 and 3).

The theory of Gaussian elimination can be summarized as follows:
In the **nonsingular** case, there is a permutation matrix P that reorders the rows of A to avoid zeros in the pivot positions. In this case

(a) $Ax = b$ has a unique solution
(b) it is found by elimination with row exchanges
(c) with the rows reordered in advance. PA can be factored into LU.

In **singular** case, no reordering can produce a full set of pivots.

Caution about L

Suppose elimination subtracts row 1 from row 2, creating $\ell_{21} = 1$. Then suppose it exchanges rows 2 and 3. If that exchange is done in advance, the multiplier will change to $\ell_{31} = 1$ in $PA = LU$.

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 3 \\ 2 & 5 & 8 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 3 & 6 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 \\ 0 & 3 & 6 \\ 0 & 0 & 2 \end{pmatrix} = U.$$

With the rows exchanged, we recover LU - but now $\ell_{31} = 1$ and $\ell_{21} = 2$.

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \text{ and } PA = LU.$$

Inverse

The matrix is **invertible** if there exists a matrix B such that $BA = I$ and $AB = I$. There is at most one such B, called **the inverse of** A and denoted by A^{-1}: $A^{-1}A = I$ and $AA^{-1} = I$.

A product AB of invertible matrices has an inverse. It is found by multiplying the individual inverses in reverse order: $(AB)^{-1} = B^{-1}A^{-1}$.

Consider the equation $AA^{-1} = I$. If it is taken a column at a time, that equation determines the column of A^{-1}. The first column of the identity matrix I is the product of A and the first column of A^{-1}.

Consider a square matrix of order 3. Let x_1, x_2, x_3 be the columns of A^{-1}.

Then $Ax_1 = e_1, Ax_2 = e_2, Ax_3 = e_3$. Thus we have three systems of equations (or, in general n systems) and they all have the same coefficient matrix A.

The right sides are different, but it is possible to carry out elimination on **all systems simultaeously**. This is called the **Gauss-Jordan method**.

Instead of stopping at U and switching to back-substitution, it continues by subtracting multiples of a row, from the rows above. It produces zeros above the diagonal as well as below, and when it reaches the identity matrix we have found A^{-1}.

The example keeps all three columns e_1, e_2, e_3, and operates on rows of length six: $[A\ e_1\ e_2\ e_3]$ becomes

$$\begin{pmatrix} 2 & 1 & 1 & 1 & 0 & 0 \\ 4 & -6 & 0 & 0 & 1 & 0 \\ -2 & 7 & 2 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & -8 & -2 & -2 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{pmatrix} = [U\ L^{-1}]$$

The first half of elimination has gone from A to U, and now the second half will go from U to I. Creating zeros above the pivots in the matrix, we reach A^{-1}: The matrix $[U\ L^{-1}] = \begin{pmatrix} 2 & 1 & 0 & 2 & -1 & -1 \\ 0 & -8 & 0 & -4 & 3 & 2 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{pmatrix}$

becomes

$$\begin{pmatrix} 1 & 0 & 0 & \frac{12}{16} & \frac{-5}{16} & \frac{-6}{16} \\ 0 & 1 & 0 & \frac{4}{8} & \frac{-3}{8} & \frac{-2}{8} \\ 0 & 0 & 1 & -1 & 1 & 1 \end{pmatrix} = [I\ A^{-1}]$$

At the last step, we divided through by the pivots. The coefficient matrix in the left half became the identity. Since A went to I, the same operations on the right half must have carried I into A^{-1}.

Therefore we have computed the inverse. The final operation count for computing A^{-1} is $n^3/6 + n^3/3 + n(n^2/2) = n^3$.

Transpose

The transpose of a lower triangular matrix is upper triangular. The transpose of A^T brings us back to A.

If we add two matrices A and B and then transpose the result is the same as first transposing and then adding: $(A+B)^T = A^T + B^T$. Also, $(AB)^T = B^T A^T$ and $(A^{-1})^T = (A^T)^{-1}$.

A special class of matrices, probably the most important class of all: A **symmetric matrix** is a matrix which equals its own transpose; $A^T = A$. The matric is necessarily square, and each entry on one side of the diagonal equals its "mirror image" on the other side $a_{ij} = a_{ji}$.

If A is symmetric, then A^{-1} is symmetric (if A^{-1} exits). Symmetric matrices appear in every subject whose laws are fair. "Each action has an equal and opposite reaction", and the entry which gives the action of i onto j is matched by the action of j onto i: The work of elimination is cut essientially in last by symmetry, from $n^3/2$ to $n^3/6$.

LDU **Factorization for Symmetric Matrices.** If A is symmetric, and if it can be factored into $A = LDU$ without row exchanges to destroy the symmetry, then the upper triangular U is the transpose of the lower triangular L. The factorization becomes $A = LDL^T$.

References

- S. Kumaresan, "*Linear Algebra - A Geometric Approach*", PHI Learning Pvt. Ltd., 2011.
- Gilbert Strang, "*Linear Algebra and its Applications*", Cengage Learning, New Delhi, 2006.