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Advanced Linear Algebra (MA 409)
Problem Sheet - 20

Diagonalizability

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1. Label the following statements as true or false.

   (a) Any linear operator on an n-dimensional vector space that has fewer than n distinct eigenvalues is not diagonalizable.

   (b) Two distinct eigenvectors corresponding to the same eigenvalue are always linearly dependent.

   (c) If λ is an eigenvalue of a linear operator T, then each vector in E_λ is an eigenvector of T.

   (d) If λ_1 and λ_2 are distinct eigenvalues of a linear operator T, then $E_{\lambda_1} \cap E_{\lambda_2} = \{0\}$.

   (e) Let $A \in M_{n\times n}(F)$ and $\beta = \{v_1, v_2, \dots, v_n\}$ be an ordered basis for F^n consisting of eigenvectors of A. If Q is the $n \times n$ matrix whose jth column is $v_j (1 \le j \le n)$, then $Q^{-1}AQ$ is a diagonal matrix.

   (f) A linear operator T on a finite-dimensional vector space is diagonalizable if and only if the multiplicity of each eigenvalue λ equals the dimension of E_λ.

   (g) Every diagonalizable linear operator on a nonzero vector space has at least one eigenvalue.

   The following two items relate to the optional subsection on direct sums.

   (h) If a vector space is the direct sum of subspaces $W_1, W_2, \dots, W_k$, then $W_i \cap W_j = \{0\}$ for $i \neq j$.

   (i) If

   $$V = \sum_{i=1}^{k} W_i \quad \text{and} \quad W_i \cap W_j = \{0\} \quad \text{for } i \neq j,$$

   then $V = W_1 \oplus W_2 \oplus \cdots \oplus W_k$.

2. For each of the following matrices $A \in M_{n\times n}(\mathbb{R})$, test A for diagonalizability, and if A is diagonalizable, find an invertible matrix Q and a diagonal matrix D such that $Q^{-1}AQ = D$.

   a) $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ b) $\begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$ c) $\begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix}$

   d) $\begin{pmatrix} 7 & -4 & 0 \\ 8 & -5 & 0 \\ 6 & -6 & 3 \end{pmatrix}$ e) $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}$ f) $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{pmatrix}$

   g) $\begin{pmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ -1 & -1 & 1 \end{pmatrix}$

3. For each of the following linear operators T on a vector space V, test T for diagonalizability, and if T is diagonalizable, find a basis β for V such that $[T]_\beta$ is a diagonal matrix.

   (a) $V = P_3(\mathbb{R})$ and T is defined by $T(f(x)) = f'(x) + f''(x)$, respectively.

   (b) $V = P_2(\mathbb{R})$ and T is defined by $T(ax^2 + bx + c) = cx^2 + bx + a$.

   (c) $V = \mathbb{R}^3$ and T is defined by

$$T\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} a_2 \\ -a_1 \\ 2a_3 \end{pmatrix}.$$

   (d) $V = P_2(\mathbb{R})$ and T is defined by $T(f(x)) = f(0) + f(1)(x + x^2)$.

   (e) $V = \mathbb{C}^2$ and T is defined by $T(z, w) = (z + iw, iz + w)$.

   (f) $V = M_{2\times 2}(\mathbb{R})$ and T is defined by $T(A) = A^t$.

4. Prove that if $A \in M_{n\times n}(F)$ has n distinct eigenvalues, then A is diagonalizable.

5. Prove that if $A \in M_{n\times n}(F)$ is diagonalizable, then the characteristic polynomial of A splits.

6. For

$$A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} \in M_{2\times 2}(\mathbb{R}),$$

   find an expression for A^n, where n is an arbitrary positive integer.

7. Suppose that $A \in M_{n\times n}(F)$ has two distinct eigenvalues, λ_1 and λ_2, and that $\dim(E_{\lambda_1}) = n - 1$. Prove that A is diagonalizable.

8. Let T be a linear operator on a finite-dimensional vector space V, and suppose there exists an ordered basis β for V such that $[T]_\beta$ is an upper triangular matrix.

   (a) Prove that the characteristic polynomial for T splits.

   (b) State and prove an analogous result for matrices.

9. Let T be a linear operator on a finite-dimensional vector space V with the distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$ and corresponding multiplicities $m_1, m_2, \ldots, m_k$. Suppose that β is a basis for V such that $[T]_\beta$ is an upper triangular matrix. Prove that the diagonal entries of $[T]_\beta$ are $\lambda_1, \lambda_2, \ldots, \lambda_k$ and that each λ_i occurs m_i times $(1 \le i \le k)$.

10. Let A be an $n \times n$ matrix that is similar to an upper triangular matrix and has the distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$ with corresponding multiplicities $m_1, m_2, \ldots, m_k$. Prove the following statements.

    (a) $tr(A) = \sum_{i=1}^{k} m_i \lambda_i$

    (b) $\det(A) = (\lambda_1)^{m_1}(\lambda_2)^{m_2}\ldots(\lambda_k)^{m_k}$.

11. Let T be an invertible linear operator on a finite-dimensional vector space V.

    (a) Recall that for any eigenvalue λ of T, λ^{-1} is an eigenvalue of T^{-1}. Prove that the eigenspace of T corresponding to λ is the same as the eigenspace of T^{-1} corresponding to λ^{-1}.

    (b) Prove that if T is diagonalizable, then T^{-1} is diagonalizable.

12. Let $A \in M_{n\times n}(F)$. Recall that A and A^t have the same characteristic polynomial and hence share the same eigenvalues with the same multiplicities. For any eigenvalue λ of A and A^t, let E_λ and E'_λ denote the corresponding eigenspaces for A and A^t, respectively.

    (a) Show by way of example that for a given common eigenvalue, these two eigenspaces need not be the same.

    (b) Prove that for any eigenvalue λ, $\dim(E_\lambda) = \dim(E'_\lambda)$.

    (c) Prove that if A is diagonalizable, then A^t is also diagonalizable.

13. Find the general solution to each system of differential equations.

    a) $x' = x + y$
    $y' = 3x - y$

    b) $x_1' = 8x_1 + 10x_2$
    $x_2' = -5x_1 - 7x_2$

    c) $x_1' = x_1 + x_3$
    $x_2' = x_2 + x_3$
    $x_3' = 2x_3$

14. Let

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

be the coefficient matrix of the system of differential equations

$$\begin{aligned} x_1' &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ x_2' &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ &\vdots \\ x_n' &= a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n. \end{aligned}$$

Suppose that A is diagonalizable and that the distinct eigenvalues of A are $\lambda_1, \lambda_2, \ldots, \lambda_k$. Prove that a differentiable function $x : \mathbb{R} \to \mathbb{R}^n$ is a solution to the system if and only if x is of the form

$$x(t) = e^{\lambda_1 t}z_1 + e^{\lambda_2 t}z_2 + \cdots + e^{\lambda_k t}z_k,$$

where $z_i \in E_{\lambda_i}$ for $i = 1, 2, \ldots, k$. Use this result to prove that the set of solutions to the system is an n-dimensional real vector space.

15. Let $C \in M_{m\times n}(\mathbb{R})$, and let Y be an $n \times p$ matrix of differentiable functions. Prove $(CY)' = CY'$, where $(Y')_{ij} = Y'_{ij}$ for all i, j.

Exercises 16 through 18 are concerned with simultaneous diagonalization.

Definitions. Two linear operators T and U on a finite-dimensional vector space V are called **simultaneously diagonalizable** if there exists an ordered basis β for V such that both $[T]_\beta$ and $[U]_\beta$ are diagonal matrices. Similarly, $A, B \in M_{n\times n}(F)$ are called **simultaneously diagonalizable** if there exists an invertible matrix $Q \in M_{n\times n}(F)$ such that both $Q^{-1}AQ$ and $Q^{-1}BQ$ are diagonal matrices.

16. (a) Prove that if T and U are simultaneously diagonalizable linear operators on a finite-dimensional vector space V, then the matrices $[T]_\beta$ and $[U]_\beta$ are simultaneously diagonalizable for any ordered basis β.

    (b) Prove that if A and B are simultaneously diagonalizable matrices, then L_A and L_B are simultaneously diagonalizable linear operators.

17. (a) Prove that if T and U are simultaneously diagonalizable operators, then T and U commute (i.e., $TU = UT$).

    (b) Show that if A and B are simultaneously diagonalizable matrices, then A and B commute.

18. Let T be a diagonalizable linear operator on a finite-dimensional vector space, and let m be any positive integer. Prove that T and T^m are simultaneously diagonalizable.

19. Let $W_1, W_2, \ldots, W_k$ be subspaces of a finite-dimensional vector space V such that

$$\sum_{i=1}^{k} W_i = V.$$

Prove that V is the direct sum of $W_1, W_2, \ldots, W_k$ if and only if

$$\dim(V) = \sum_{i=1}^{k} \dim(W_i).$$

20. Let V be a finite-dimensional vector space with a basis β, and let $\beta_1, \beta_2, \ldots, \beta_k$ be a partition of β (i.e., $\beta_1, \beta_2, \ldots, \beta_k$ are subsets of β such that $\beta = \beta_1 \cup \beta_2 \cup \cdots \cup \beta_k$ and $\beta_i \cap \beta_j = \varnothing$ if $i \neq j$). Prove that $V = span(\beta_1) \oplus span(\beta_2) \oplus \cdots \oplus span(\beta_k)$.

21. Let T be a linear operator on a finite-dimensional vector space V, and suppose that the distinct eigenvalues of T are $\lambda_1, \lambda_2, \ldots, \lambda_k$. Prove that

$$span(\{x \in V : x \text{ is an eigenvector of } T\}) = E_{\lambda_1} \oplus E_{\lambda_2} \oplus \cdots \oplus E_{\lambda_k}.$$

22. Let $W_1, W_2, K_1, K_2, \ldots, K_p, M_1, M_2, \ldots, M_q$ be subspaces of a vector space V such that $W_1 = K_1 \oplus K_2 \oplus \cdots \oplus K_p$ and $W_2 = M_1 \oplus M_2 \oplus \cdots \oplus M_q$. Prove that if $W_1 \cap W_2 = \{0\}$, then

$$W_1 + W_2 = W_1 \oplus W_2 = K_1 \oplus K_2 \oplus \cdots \oplus K_p \oplus M_1 \oplus M_2 \oplus \cdots \oplus M_q.$$

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