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Advanced Linear Algebra (MA 409)
Problem Sheet - 12

The Rank of a Matrix and Matrix Inverses

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1. Label the following statements as true or false.
   (a) The rank of a matrix is equal to the number of its nonzero columns.
   (b) The product of two matrices always has rank equal to the lesser of the ranks of the two matrices.
   (c) The $m \times n$ zero matrix is the only $m \times n$ matrix having rank 0.
   (d) Elementary row operations preserve rank.
   (e) Elementary column operations do not necessarily preserve rank.
   (f) The rank of a matrix is equal to the maximum number of linearly independent rows in the matrix.
   (g) The inverse of a matrix can be computed exclusively by means of elementary row operations.
   (h) The rank of an $n \times n$ matrix is at most n.
   (i) An $n \times n$ matrix having rank n is invertible.

2. Find the rank of the following matrices.

   a) $\begin{pmatrix} 1 & 1 & 0 \\ 2 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ b) $\begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \end{pmatrix}$

   c) $\begin{pmatrix} 1 & 2 & 3 & 1 & 1 \\ 1 & 4 & 0 & 1 & 2 \\ 0 & 2 & -3 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$ d) $\begin{pmatrix} 1 & 1 & 0 & 1 \\ 2 & 2 & 0 & 2 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \end{pmatrix}$

3. Prove that for any $m \times n$ matrix A, $rank(A) = 0$ if and only if A is the zero matrix.

4. Use elementary row and column operations to transform each of the following matrices into a matrix D satisfying the conditions of Theorem 3.6, and then determine the rank of each matrix.

   (a) $\begin{pmatrix} 1 & 1 & 1 & 2 \\ 2 & 0 & -1 & 2 \\ 1 & 1 & 1 & 2 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & 1 \\ -1 & 2 \\ 2 & 1 \end{pmatrix}$

5. For each of the following matrices, compute the rank and the inverse if it exists.

   a) $\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$ b) $\begin{pmatrix} 0 & -2 & 4 \\ 1 & 1 & -1 \\ 2 & 4 & -5 \end{pmatrix}$

c) $\begin{pmatrix} 1 & 2 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

d) $\begin{pmatrix} 1 & 2 & 1 & 0 \\ 2 & 5 & 5 & 1 \\ -2 & -3 & 0 & 3 \\ 3 & 4 & -2 & -3 \end{pmatrix}$

e) $\begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & -1 & 2 \\ 2 & 0 & 1 & 0 \\ 0 & -1 & 1 & -3 \end{pmatrix}$

6. For each of the following linear transformations T, determine whether T is invertible, and compute T^{-1} if it exists.

   (a) $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ defined by $T(f(x)) = f''(x) + 2f'(x) - f(x)$.

   (b) $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ defined by $T(f(x)) = (x+1)f'(x)$.

   (c) $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

   $$T(a_1, a_2, a_3) = (a_1 + 2a_2 + a_3, -a_1 + a_2 + 2a_3, a_1 + a_3).$$

   (d) $T : \mathbb{R}^3 \to P_2(\mathbb{R})$ defined by

   $$T(a_1, a_2, a_3) = (a_1 + a_2 + a_3) + (a_1 - a_2 + a_3)x + a_1x^2.$$

   (e) $T : P_2(\mathbb{R}) \to \mathbb{R}^3$ defined by $T(f(x)) = (f(-1), f(0), f(1))$.

   (f) $T : M_{2\times 2}(\mathbb{R}) \to \mathbb{R}^4$ defined by

   $$T(A) = (tr(A), tr(A^t), tr(EA), tr(AE)),$$

   where

   $$E = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

7. Express the invertible matrix

   $$\begin{pmatrix} 1 & 2 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

   as a product of elementary matrices.

8. Let A be an $m \times n$ matrix. Prove that if c is any nonzero scalar, then $rank(cA) = rank(A)$.

9. Let

   $$B = \left(\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & B' & \\ 0 & & & \end{array}\right),$$

   where B' is an $m \times n$ submatrix of B. Prove that if $rank(B) = r$, then $rank(B') = r - 1$.

10. Let B' and D' be $m \times n$ matrices, and let B and D be $(m+1) \times (n+1)$ matrices respectively defined by

$$B = \left(\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & B' & \\ 0 & & & \end{array}\right) \quad \text{and} \quad D = \left(\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & D' & \\ 0 & & & \end{array}\right).$$

Prove that if B' can be transformed into D' by an elementary row [column] operation, then B can be transformed into D by an elementary row [column] operation.

11. Let $T, U : V \to W$ be linear transformations.

    (a) Prove that $R(T+U) \subseteq R(T) + R(U)$. (See the definition of the sum of subsets of a vector space on page 22.)

    (b) Prove that if W is finite-dimensional, then $rank(T+U) \leq rank(T) + rank(U)$.

    (c) Deduce from (b) that $rank(A+B) \leq rank(A) + rank(B)$ for any $m \times n$ matrices A and B.

12. Suppose that A and B are matrices having n rows. Prove that $M(A|B) = (MA|MB)$ for any $m \times n$ matrix M.

13. Prove that if B is a 3×1 matrix and C is a 1×3 matrix, then the 3×3 matrix BC has rank at most 1. Conversely, show that if A is any 3×3 matrix having rank 1, then there exist a 3×1 matrix B and a 1×3 matrix C such that $A = BC$.

14. Let A be an $m \times n$ matrix and B be an $n \times p$ matrix. Prove that AB can be written as a sum of n matrices of rank at most one.

15. Let A be an $m \times n$ matrix with rank m and B be an $n \times p$ matrix with rank n. Determine the rank of AB. Justify your answer.

16. Let

$$A = \begin{pmatrix} 1 & 0 & -1 & 2 & 1 \\ -1 & 1 & 3 & -1 & 0 \\ -2 & 1 & 4 & -1 & 3 \\ 3 & -1 & -5 & 1 & -6 \end{pmatrix}.$$

    (a) Find a 5×5 matrix M with rank 2 such that $AM = O$, where O is the 4×5 zero matrix.

    (b) Suppose that B is a 5×5 matrix such that $AB = O$. Prove that $rank(B) \leq 2$.

17. Let A be an $m \times n$ matrix with rank m. Prove that there exists an $n \times m$ matrix B such that $AB = I_m$.

18. Let B be an $n \times m$ matrix with rank m. Prove that there exists an $m \times n$ matrix A such that $AB = I_m$.

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