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Advanced Linear Algebra (MA 409)
Problem Sheet - 28

The Singular Value Decomposition and the Pseudoinverse

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1. Label the following statements as true or false.
   (a) The singular values of any linear operator on a finite-dimensional vector space are also eigenvalues of the operator.
   (b) The singular values of any matrix A are the eigenvalues of A^*A.
   (c) For any matrix A and any scalar c, if σ is a singular value of A, then $|c|\sigma$ is a singular value of cA.
   (d) The singular values of any linear operator are nonnegative.
   (e) If λ is an eigenvalue of a self-adjoint matrix A, then λ is a singular value of A.
   (f) For any $m \times n$ matrix A and any $b \in F^n$, the vector $A^\dagger b$ is a solution to $Ax = b$.
   (g) The pseudoinverse of any linear operator exists even if the operator is not invertible.

2. Let $T : V \to W$ be a linear transformation of rank r, where V and W are finite-dimensional inner product spaces. In each of the following, find orthonormal bases $\{v_1, v_2, \ldots, v_n\}$ for V and $\{u_1, u_2, \ldots, u_m\}$ for W, and the nonzero singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r$ of T such that $T(v_i) = \sigma_i u_i$ for $1 \leq i \leq r$.
   (a) $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(x_1, x_2) = (x_1, x_1 + x_2, x_1 - x_2)$
   (b) Let $P_2(\mathbb{R})$ and $P_1(\mathbb{R})$ be the polynomial spaces with inner product defined by
   $$\langle f(x), g(x) \rangle = \int_{-1}^{1} f(t)g(t)\, dt.$$
   Let $T : P_2(\mathbb{R}) \to P_1(\mathbb{R})$ be the linear transformation defined by $T(f(x)) = f''(x)$.
   (c) Let $V = W = span(\{1, \sin x, \cos x\})$ with the inner product defined by
   $$\langle f, g \rangle = \int_{0}^{2\pi} f(t)g(t)\, dt$$
   and T is defined by $T(f) = f' + 2f$
   (d) $T : \mathbb{C}^2 \to \mathbb{C}^2$ defined by $T(z_1, z_2) = ((1-i)z_2, (1+i)z_1 + z_2)$

3. Find a singular value decomposition for each of the following matrices.

a) $\begin{pmatrix} 1 & 1 \\ 1 & 1 \\ -1 & -1 \end{pmatrix}$ b) $\begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & -1 \end{pmatrix}$ c) $\begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix}$

d) $\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}$ e) $\begin{pmatrix} 1+i & 1 \\ 1-i & -i \end{pmatrix}$ f) $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & -2 & 1 \\ 1 & -1 & 1 & 1 \end{pmatrix}$

4. Find a polar decomposition for each of the following matrices.

a) $\begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix}$ b) $\begin{pmatrix} 20 & 4 & 0 \\ 0 & 0 & 1 \\ 4 & 20 & 0 \end{pmatrix}$

5. Find an explicit formula for each of the following expressions.

(a) $T^{\dagger}(x_1, x_2, x_3)$, where T is the linear transformation of Exercise 2a
(b) $T^{\dagger}(a + bx + cx^2)$, where T is the linear transformation of Exercise 2b
(c) $T^{\dagger}(a + b\sin x + c\cos x)$, where T is the linear transformation of Exercise 2b
(d) $T^{\dagger}(z_1, z_2)$, where T is the linear transformation of Exercise 2d

6. Use the results of Exercise 3 to find the pseudoinverse of each of the following matrices.

a) $\begin{pmatrix} 1 & 1 \\ 1 & 1 \\ -1 & -1 \end{pmatrix}$ b) $\begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & -1 \end{pmatrix}$ c) $\begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix}$

d) $\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}$ e) $\begin{pmatrix} 1+i & 1 \\ 1-i & -i \end{pmatrix}$ f) $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & -2 & 1 \\ 1 & -1 & 1 & 1 \end{pmatrix}$

7. For each of the given linear transformations $T : V \to W$,

(i) Describe the subspace Z_1 of V such that $T^{\dagger}T$ is the orthogonal projection of V on Z_1.
(ii) Describe the subspace Z_2 of W such that $TT^{\dagger}$ is the orthogonal projection of W on Z_2.

(a) T is the linear transformation of Exercise 2a
(b) T is the linear transformation of Exercise 2b
(c) T is the linear transformation of Exercise 2c
(d) T is the linear transformation of Exercise 2d

8. For each of the given systems of linear equations,

(i) If the system is consistent, find the unique solution having minimum norm.
(ii) If the system is inconsistent, find the "best approximation to a solution "having minimum norm.

(Use your answers to parts (a) and (f) of Exercise 6.)

a) $x_1 + x_2 = 1$
$x_1 + x_2 = 2$
$-x_1 + -x_2 = 0$

b) $x_1 + x_2 + x_3 + x_4 = 2$
$x_1 - 2x_3 + x_4 = -1$
$x_1 - x_2 + x_3 + x_4 = 2$

9. Let V and W be finite-dimensional inner product spaces over F, and suppose that $\{v_1, v_2, \ldots, v_n\}$ and $\{u_1, u_2, \ldots, u_m\}$ are orthonormal bases for V and W, respectively. Let $T : V \to W$ is a linear transformation of rank r, and suppose that $\sigma_1 \geq \sigma_1 \geq \cdots \geq \sigma_r > 0$ are such that

$$T(v_i) = \begin{cases} \sigma_i u_i & \text{if } 1 \leq i \leq r \\ 0 & \text{if } r < i. \end{cases}$$

(a) Prove that $\{u_1, u_2, \ldots, u_m\}$ is a set of eigenvectors of TT^* with corresponding eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_m$, where

$$\lambda_i = \begin{cases} \sigma_i^2 & \text{if } 1 \leq i \leq r \\ 0 & \text{if } r < i. \end{cases}$$

(b) Let A be an $m \times n$ matrix with real or complex entries. Prove that the nonzero singular values of A are the positive square roots of the nonzero eigenvalues of AA^*, including repetitions.

(c) Prove that TT^* and T^*T have the same nonzero eigenvalues, including repetitions.

(d) State and prove a result for matrices analogous to (c).

10. We have proved the following result : Let $T : V \to W$ be a linear transformation from a finite-dimensional vector space V to a finite-dimensional vector space W. Let β and β' be ordered bases for V, and let γ and γ' be ordered bases for W. Then prove that $[T]_{\beta'}^{\gamma'} = P^{-1}[T]_{\beta}^{\gamma}Q$, where Q is the matrix that changes β'-coordinates into β-coordinates and P is the matrix that changes γ'-coordinates into γ-coordinates.

    Use the above result to obtain another proof of the singular value decomposition theorem for matrices.

11. This exercise relates the singular values of a well-behaved linear operator or matrix to its eigenvalues.

    (a) Let T be a normal linear operator on an n-dimensional inner product space with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Prove that the singular values of T are $|\lambda_1|, |\lambda_2|, \ldots, |\lambda_n|$.

    (b) State and prove a result for matrices analogous to (a).

12. Let A be a normal matrix with an orthonormal basis of eigenvectors $\beta = \{v_1, v_2, \ldots, v_n\}$ and corresponding eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Let V be the $n \times n$ matrix whose columns are the vectors in β. Prove that for each i there is a scalar θ_i of absolute value 1 such that if U is the $n \times n$ matrix with $\theta_i v_i$ as column i and $\sum$ is the diagonal matrix such that $\sum_{ii} = |\lambda_i|$ for each i, then $U \sum V^*$ is a singular value decomposition of A.

13. Prove that if A is a positive semidefinite matrix, then the singular values of A are the same as the eigenvalues of A.

14. Prove that if A is a positive definite matrix and $A = U \sum V^*$ is a singular value decomposition of A, then $U = V$.

15. Let A be a square matrix with a polar decomposition $A = WP$.

    (a) Prove that A is normal if and only if $WP^2 = P^2W$.

    (b) Use (a) to prove that A is normal if and only if $WP = PW$.

16. Let A be a square matrix. Prove an alternate form of the polar decomposition for A : There exists a unitary matrix W and a positive semidefinite matrix P such that $A = PW$.

17. Let T and U be linear operators on $\mathbb{R}^2$ defined for all $(x_1, x_2) \in \mathbb{R}^2$ by

$$T(x_1, x_2) = (x_1, 0) \quad \text{and} \quad U(x_1, x_2) = (x_1 + x_2, 0).$$

    (a) Prove that $(UT)^\dagger \neq T^\dagger U^\dagger$.

(b) Exhibit matrices A and B such that AB is defined, but $(AB)^\dagger \neq B^\dagger A^\dagger$.

18. Let A be an $m \times n$ matrix. Prove the following results.

    (a) For any $m \times m$ unitary matrix G, $(GA)^\dagger = A^\dagger G^*$.

    (b) For any $n \times n$ unitary matrix H, $(AH)^\dagger = H^* A^\dagger$.

19. Let A be a matrix with real or complex entries. Prove the following results.

    (a) The nonzero singular values of A are the same as the nonzero singular values of A^*, which are the same as the nonzero singular values of A^t.

    (b) $(A^\dagger)^* = (A^*)^\dagger$.

    (c) $(A^\dagger)^t = (A^t)^\dagger$.

20. Let A be a square matrix such that $A^2 = O$. Prove that $(A^\dagger)^2 = O$.

21. Let V and W be finite-dimensional inner product spaces, and let $T : V \to W$ be linear. Prove the following results.

    (a) $TT^\dagger T = T$.

    (b) $T^\dagger T T^\dagger = T^\dagger$.

    (c) Both $T^\dagger T$ and $TT^\dagger$ are self-adjoint.

    The preceding three statements are called the **Penrose conditions**, and they characterize the pseudoinverse of a linear transformation as shown in Exercise 22.

22. Let V and W be finite-dimensional inner product spaces. Let $T : V \to W$ and $U : W \to V$ be linear transformations such that $TUT = T$, $UTU = U$, and both UT and TU are self-adjoint. Prove that $U = T^\dagger$.

23. State and prove a result for matrices that is analogous to the result of Exercise 21.

24. State and prove a result for matrices that is analogous to the result of Exercise 22.

25. Let V and W be finite-dimensional inner product spaces, and let $T : V \to W$ be linear. Prove the following results

    (a) If T is one-to-one, then T^*T is invertible and $T^\dagger = (T^*T)^{-1}T^*$.

    (b) If T is onto, then TT^* is invertible and $T^\dagger = T^*(TT^*)^{-1}$.

26. Let V and W be finite-dimensional inner product spaces with orthonormal bases β and γ, respectively, and let $T : V \to W$ be linear. Prove that $([T]_\beta^\gamma)^\dagger = [T^\dagger]_\gamma^\beta$.

27. Let V and W be finite-dimensional inner product spaces, and let $T : V \to W$ be a linear transformation. Prove that $TT^\dagger$ is the orthogonal projection of W on $R(T)$.

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