Vector Spaces

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February 5, 2018

File Name : vector spaces

Dated : February 5, 2018

Overview

Prerequisite for the course "Advanced Linear Algebra" is the following:

- MA204 - Linear Algebra and Matrices
- EC224 - Mathematics for E&C Engineering
- EE243 - Linear Algebra

In the above courses, you may have applied the language of linear algebra to vectors in $\mathbb{R}^n$. Some of the key words of this language are linear combination, linear transformation, kernel, image, subspace, span, linear independence, basis, dimension, and coordinates.

Note that all these concepts can be defined in terms of sums and scalar multiples of vectors in a general setting.

Overview

Linear algebra is a branch of mathematics which treats the common properties of algebraic systems which consist of a set, together with a reasonable notion of a "linear combination" of elements in the set.

We shall define the mathematical object, called a "vector space" which is a composite object consisting of

- a field of "scalars",
- a set of "vectors",
- and two operations (addition and scalar multiplication) with certain special properties.

We discuss properties of vectors spaces and examples in two lectures.

Fields

Definition 1.

A nonemtpy set F has two operations as follows. The first operation, called **addition***, associates with each pair of elements α, β in F an element $(\alpha + \beta)$ in F ; the second operation, called* **multiplication***, associates with each pair α, β an element $\alpha\beta$ in F ; and these two operations satisfy the following* **nine rules** *of algebra given below.*

1. *Addition is commutative : $\alpha + \beta = \beta + \alpha$, for all α and β in F.*
2. *Addition is associative : $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$ for all α, β, and γ in F.*
3. *There is a unique element 0 (zero, or, additive identity) in F such that $\alpha + 0 = \alpha$, for every α in F.*
4. *To each α in F there corresponds a unique element $(-\alpha)$ in F such that $\alpha + (-\alpha) = 0$.*

Definition 2 (contd . . .).

5. *Multiplication is commutative :* $\alpha\beta = \beta\alpha$ *for all* α *and* β *in* F.
6. *Multiplication is associative :* $\alpha(\beta\gamma) = (\alpha\beta)\gamma$ *for all* α, β, *and* γ *in* F.
7. *There is a unique non-zero element* 1 *(one, or, multiplicative identity) in* F *such that* $\alpha 1 = \alpha$, *for every* α *in* F.
8. *To each non-zero* α *in* F *there corresponds a unique element* α^{-1} *(or* $1/\alpha$*) in* F *such that* $\alpha\alpha^{-1} = 1$.
9. *Multiplication distributes over addition; that is,* $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$, *for all* α, β, *and* γ *in* F.

The set F, *together with these two operations, is called a* **field**.

The set $\{0, 1\}$, the set $\mathbb{Q}$ of rational numbers, the set $\mathbb{R}$ of real numbers and the set $\mathbb{C}$ of complex numbers are fields.

Example 3.

With the usual operations of addition and multiplication, the set $\mathbb{C}$ of complex numbers is a field, as is the set $\mathbb{R}$ of real numbers.

Notation

We shall use the word "**scalar**" to represent any element from a field F.

Definition 4.

A **subfield** *S of the field F is itself a field under the operations of addition and multiplication defined for F.*

The set $\{0, 1\}$ is a subfield of $\mathbb{R}$.

We give below the definition of a vector space over a **general field**. The main reason for this is that vector spaces over finite fields are of great interest in Computer Science, Coding Theory, Combinatorics, Design of Experiments and Abstract Algebra; vector spaces over the field of rational numbers are useful in Number Theory and Design of Experiments and vector spaces over the field of complex numbers are needed for the study of eigenvalues. Thus vector spaces over fields other than $\mathbb{R}$ are useful in many contexts.

Besides, the theory of vector spaces over a general field is no more complicated than that over $\mathbb{R}$. So throughout the course, we consider vector spaces over a general field. However, many important examples of vector spaces take the field to be $\mathbb{R}$ or $\mathbb{C}$ and we would not lose much by making this assumption. In fact, for easy viualization, one can take the field to be $\mathbb{R}$ in most cases.

Definition 5.

A **vector space** *(or* **linear space***) consists of the following :*

1. *a field F of* **scalars** *;*
2. *a set V of objects, called* **vectors** *(reason for calling "vectors" will be discussed later) ;*
3. *a rule (or operation), called* **vector addition***, which associates with each pair of vectors x, y in V a vector $x + y$ in V, called the sum of x and y, in such a way that*
   (a) *addition is commutative, $x + y = y + x$;*
   (b) *addition is associative, $x + (y + z) = (x + y) + z$;*
   (c) *there is a unique vector 0 in V, called the* **zero vector***, such that $x + 0 = x$ for all x in V ;*
   (d) *for each vector x in V there is a unique vector $-x$ in V such that $x + (-x) = 0$ ($-x$ is called the negative of x) ;*

Definition 6 (contd . . .).

4. *a rule (or operation), called* **scalar multiplication**, *which associates with each scalar* α *in* F *and vector* x *in* V *a vector* αx *in* V, *called the* **product** *of* α *and* x, *in such a way that*
   (a) $1x = x$ *for every* x *in* V ;
   (b) $(\alpha\beta)x = \alpha(\beta x)$;
   (c) $\alpha(x + y) = \alpha x + \alpha y$;
   (d) $(\alpha + \beta)x = \alpha x + \beta x$.

F *itself is called the* **base field** *or* **ground field** *of the vector space.*

We adopt the following standard notation : $x + (-y)$ is written as $x - y$ for all $x, y \in V$ and for $\alpha \in F$ and $x \in V$ we write αx for $\alpha.x$.

A **real** (respectively **complex**) vector space is a vector space over $\mathbb{R}$ (respectively $\mathbb{C}$). Note that $\mathbb{C}$ with the usual addition and multiplication of a real number with a complex number, can be considered to be a real vector space.

Notation

We will normally use lower case Roman letters (like x, y, x_1) to denote vectors and lower case Greek letters (like α, β, ξ_1) to denote the scalars.

These axioms were established by Italian mathematician **Giuseppe Peano** (1858-1932) in his Calcolo Geometrico of 1888.

Peano calls V a "linear space."

* picture taken from Google website

The n-tuple space F^n:

Example 7.

Let F be any field and let V be the set of all n-tuples

$$x = (\alpha_1, \alpha_2, \ldots, \alpha_n)$$

of scalars α_i in F. Then V is a vector space over F with respect to **coordinatewise** *addition and scalar multiplication, and it is denoted by F^n.*
By "the vector space F^n," we mean F^n over F.

The sequence space F^∞:

Example 8.

Let F be any field and let V be the set of all sequences

$$x = (\alpha_1, \alpha_2, \ldots)$$

of scalars α_i in F. Then V is a vector space over F with respect to **coordinatewise** *addition and scalar multiplication, and it is denoted by F^∞.*

The n-tuple space F^n:

Example 9.

Let F be any field and let m and n be the integers. The set $F^{m\times n}$ of all $m \times n$ matrices is a vector space over F with respect to **componentwise** *addition and scalar multiplication.*

Note that $F^{1\times n} = F$.

The zero vector is the zero matrix, whose entries are all zero.

We shall see that this space is almost the same as F^{mn}. The "mn" components are arranged in a rectangle instead of a column.

Notation :

When we say F^n is a vector space, it is understood that F^n is a vector space over F.

If it is desirable to specify the field we shall mention the vector space with the field (for instance, refer Example 14).

Example 10.

Let F be any field and let S be any **nonemtpy set**. *The set V of all functions from the set S into F is a vector space over F with respect to* **pointwise** *addition and scalar multiplication defined by*

$$(f+g)(s) = f(s) + g(s) \quad ; \quad (\alpha f)(s) = \alpha f(s).$$

It is denoted by F^S.

We observe that to define sum of two vectors f and g in V, we use "sum of scalars" in F. Similarly, to define scalar multiplication in V, we use "multiplication of scalars" in F . There is nothing special about the field F. One can define "addition" and "scalar multiplicaiton" over the set of all functions from a nonempty set S into a vector space, with the help of "addition" and "scalar multiplication" defined on V.

Exercise 11.

Show that the preceding Examples (7), (8) and (9) are special cases of the Example (10).
[Hint : An n-tuple of elements in F may be regarded as a function from the set S of integers $1, 2, \ldots, n$ into F. Similarly, an $m \times n$ matrix over the field F is a function from the set S of pairs of integers $(i, j),\ 1 \leq i \leq m,\ 1 \leq j \leq n$, into the field F.]

Example 12.

Let X be a non-empty set. Let $V = F(X, \mathbb{R}) = \{f : X \to \mathbb{R}\}$ be the set of real-valued functions on the set X. Then V is a vector space over $\mathbb{R}$ (by Example (10).
Let $F_0(X, \mathbb{R})$ denote the set of functions from X to $\mathbb{R}$ such that the set $\{x \in X : f(x) \neq 0\}$ is finite (this set may depend on f). Thus, $f \in F_0(X, \mathbb{R})$ if and only if $f(x) = 0$ except for finitely many $x \in X$. Clearly, $F_0(X, \mathbb{R})$ is a subset of $F(X, \mathbb{R})$.

The space of polynomial functions over a field F:

Example 13.

Let F be any field and let V be the set of all functions f from F into F which have a rule of the form

$$f(x) = c_0 + c_1 x + \cdots + c_n x^n$$

where $c_0, c_1, \ldots, c_n$ are fixed scalars in F (independent of x).

A function of this type is called a **polynomial function** *on F.*

V is a vector space over F with respect to **pointwise** *addition and scalar multiplication.*

The set of all functions from $\mathbb{R}$ to $\mathbb{R}$ is a vector space with respect to **pointwise** addition and scalar multiplication and is denoted by $F(\mathbb{R}, \mathbb{R})$. The zero vector in the space is the zero function.

The following example illustrates that **the same set of vectors may be part of number of distinct vector spaces.**

Example 14.

The field $\mathbb{C}$ of complex numbers may be regarded as a vector space over the field $\mathbb{R}$ of real numbers. The vector space $\mathbb{C}^n$ over $\mathbb{R}$ is quite different from the space $\mathbb{C}^n$ (over $\mathbb{C}$) and the space $\mathbb{R}^n$ (over $\mathbb{R}$).

Exercises 15.

(a) *Let V denote the set of all polynomials exactly of degree n. Is it a vector space under the usual addition and scalar multiplication of polynomials ?*

(b) *Let V be the set of all solutions of a system of m homogeneous linear equations in n variables with real coefficients. Is V a vector space over $\mathbb{R}$ under obvious operations ?*

Example 16.

Let Ω be a fixed non-empty set and let V be the set of all subsets of Ω, usually known as the **power set** *of Ω.*

Vectors are the elements of V, i.e., subsets of Ω.

We define the sum of two vectors A and B to be their symmetric difference

$$A \Delta B = (A - B) \cup (B - A).$$

Here we consider the field consists only the scalars 0 and 1. Now we define the scalar multiple αA to be A if $\alpha = 1$ and $\emptyset$ (the emtpy set, or, the null set) if $\alpha = 0$.

The power set of Ω forms a vector space over $F = \{0, 1\}$ with the addition and scalar multiplication defined above.

Example 17 ($\mathbb{R}$ over $\mathbb{Q}$).

$\mathbb{R}$ *with usual addition and multiplication is a vector space over the field of rational numbers* $\mathbb{Q}$. *The zero vector here is the real number* 0 *and the negative of* x *is the real number* $-x$.

Exercises 18.

(a) *A sum involving a number of vectors is* **independent of the way** *in which these vectors are combined and associated : Prove that if V is a vector space over the field F, verify that*

$$(x_1 + x_2) + (x_3 + x_4) = [x_2 + (x_3 + x_1)] + x_4$$

for all vectors x_1, x_2, x_3 and x_4 in V.
Such a sum may be written without confusion as $x_1 + x_2 + x_3 + x_4$.

(b) *Let V be the set of all pairs (α, β) of real numbers, and let F be the field of real numbers. Define*

$$\begin{aligned}(\alpha, \beta) + (\alpha_1, \beta_1) &= (\alpha + \alpha_1, \beta + \beta_1)\\ c(\alpha, \beta) &= (c\alpha, c\beta).\end{aligned}$$

Is V, with these operations, a vector space over the field of real numbers?

Exercises 19.

(a) *On $\mathbb{R}^n$, define two operations*

$$\begin{aligned} x \oplus y &= x - y \\ \alpha.x &= -\alpha x. \end{aligned}$$

The operations on the right are the usual ones. Which of the axioms for a vector space are satisfied by $(\mathbb{R}^n, \oplus, .)$?

(b) *Let V be the set of pairs (α, β) of real numbers and let F be the field of real numbers. Define*

$$\begin{aligned} (\alpha, \beta) + (\alpha_1, \beta_1) &= (\alpha + \alpha_1, 0) \\ c(\alpha, \beta) &= (c\alpha, 0). \end{aligned}$$

Is V, with these operations, a vector space ?

Exercises 20.

(a) *Let V be the set of all complex-valued functions f on the real line such that (for all t in $\mathbb{R}$)*

$$f(-t) = \overline{f(t)}.$$

The bar denotes complex conjugation. Show that V, with the operations

$$\begin{aligned}(f+g)(t) &= f(t)+g(t)\\ (cf)(t) &= cf(t)\end{aligned}$$

is a vector space over the field of real numbers. Give an example of a function in V which is not real-valued.

Exercises 21.

(a) *Construct a subset of the xy-plane $\mathbb{R}^2$ that is*

  (i) *closed under vector addition and subtraction, but not scalar multiplication.*

  (ii) *closed under scalar multiplication but not under vector addition.*

(b) *Let c be the set of all convergent real sequences. Prove that c is a subset of $\mathbb{R}^\infty$ (the set of all real sequences) and c is a vector space with respect to coordinatewise operations.*

(c) *Let c_0 be the set of null sequences (sequences converging to 0). Prove that $c_0 \subseteq c \subseteq \mathbb{R}^\infty$ and c_0 is a vector space with respect to coordinatewise operations.*

Exercises 22.

(a) *Let $C([a,b])$ be the set of all real-valued continuous functions on $[a,b]$. This is a subset of $F([a,b],\mathbb{R})$. Show that $C([a,b])$ is a vector space over $\mathbb{R}$.*

(b) *Show that the set $D([0,1])$ of differentiable functions on $[0,1]$ is a subset of $C([0,1])$ and it is a vector space under pointwise operations.*

(c) *Show that the set $R([a,b])$ of all Riemann integrable functions on $[a,b]$ is a subset of $F([a,b],\mathbb{R})$ and is a vector space.*

(d) *Recall that a function $f:\mathbb{R}\to\mathbb{R}$ is called* **even** *(respectively* **odd***) if $f(-x)=f(x)$ for all $x\in\mathbb{R}$ (respectively $f(-x)=-f(x)$ for every $x\in\mathbb{R}$). Let $F_+(\mathbb{R},\mathbb{R})$ (respectively, $F_-(\mathbb{R},\mathbb{R})$) denote the set of even (respectively odd) functions from $\mathbb{R}$ to $\mathbb{R}$. Are they vector spaces under the obvious definitions?*

Exercises 23.

Let V be a vector space. Prove the following simple facts.

(a) $0.x = \alpha.0 = 0$;

(b) $-x$ *is unique for a given* x *in* V ;

(c) $x + y = x + z \implies y = z;$

(d) $x + x = x \implies x = 0$;

(e) *If* α *is a scalar and* x *a vector such that* $\alpha x = 0$, *then either* α *is the zero scalar or* x *is the zero vector* ;

(f) *If* x *is any vector in* V, *then*

$$(-1)x = -x.$$

Definition 24.

A vector y in V is said to be a **linear combination** *of the vectors $x_1, x_2, \ldots, x_n$ in V provided there exist scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$ in F such that*

$$y = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n = \sum_{i=1}^{n} \alpha_i x_i.$$

Using the associative property of vector addition and distributive properties of scalar multiplication, we get

$$\sum_{i=1}^{n} \alpha_i x_i + \sum_{i=1}^{n} \beta_i x_i = \sum_{i=1}^{n} (\alpha_i + \beta_i) x_i$$

$$c \sum_{i=1}^{n} \alpha_i x_i = \sum_{i=1}^{n} (c\alpha_i) x_i.$$

Linear Combination

A linear combination from a non-empty set A of vectors is a linear combination of **finitely** many vectors belonging to A. As a matter of convention, we define 0 to be the linear combination from the empty set.

We note that, whereas an expression $\sum_{i=1}^{n} \alpha_i x_i$ determines a unique vector, a vector may have different representations in the form $\sum_{i=1}^{n} \alpha_i x_i$.

Exercise 25.

If $\mathbb{C}$ is the field of complex numbers, which vectors in $\mathbb{C}^3$ are linear combinations of $(1, 0, -1), (0, 1, 1)$, and $(1, 1, 1)$?

Exercise 26.

Let $A = \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix}$. Show that $A^2 = \begin{pmatrix} 2 & 3 \\ 6 & 11 \end{pmatrix}$ is a linear combination of A and I_2 (the identity matrix of order 2).

Certain parts of linear algebra are intimately related to geometry.

We shall discuss now the origin of the "vector space".

A vector is usually defined as a directed line segment PQ, from a point P in the space to another point Q.

The vectors are determined by their length and direction. The directed line segment PQ is the "**arrow**" from P to Q. Thus one must identify two directed line segments if they have the same length and the same direction.

Certain parts of linear algebra are intimately related to geometry.

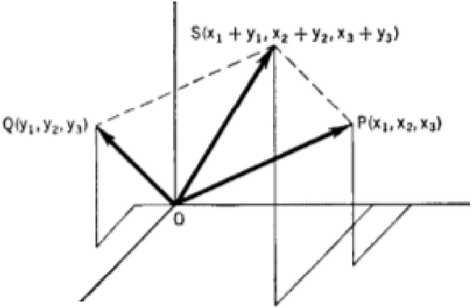
Let us consider the vector space $\mathbb{R}^3$. The directed line segment PQ, from the point $P = (x_1, x_2, x_3)$ to the point $Q = (y_1, y_2, y_3)$, has the same length and direction as the directed line segment from the origin $O = (0, 0, 0)$ to the point $(y_1 - x_1, y_2 - x_2, y_3 - x_3)$.

Furthermore, this is the only segment **emanating from the origin** which has the same length and direction as PQ. Thus, if one agrees to treat only vectors which emanate from the origin, there is **exactly one vector** associated with each given length and direction.

The vector OP, from the origin to $P = (x_1, x_2, x_3)$, is completely determined by P, and it is therefore possible to identify this vector with the point P. That is, each arrow OP can be represented by the point P once the point O which is taken to be the origin.

Certain parts of linear algebra are intimately related to geometry.

Given points $P = (x_1, x_2, x_3)$ and $Q = (y_1, y_2, y_3)$, the definition of the sum of the vectors OP and OQ can be given geometrically. If the vectors are not parallel, then the segments OP and OQ determine a plane and these segments are two of the edges of a parallelogram in that plane.



Certain parts of linear algebra are intimately related to geometry.

One diagonal of this parallelogram extends from O to a point S, and the sum of OP and OQ is defined to be the vector OS.

The coordinates of the point S are $(x_1 + y_1, x_2 + y_2, x_3 + y_3)$ and hence this geometrical definition of vector addition is equivalent to the algebraic definition of Example (8).

Scalar multiplication has a simpler geometric interpretation.

If c is a real number, then the product of c and the vector OP is the vector from the origin with length $|c|$ times the length of OP and a direction which agrees with the direction of OP is $c > 0$, and which is opposite to the direction of OP if $c < 0$.

Certain parts of linear algebra are intimately related to geometry.

In Physics we learn that a force applied at a point O has both magnitude and direction. It is represented by an arrow OP, where the length OP represents the magnitude and O to P the direction of the force. If we now apply another force OQ at the point O, the resultant (also called the sum) of the two forces is obtained by the parallelogram law : it is OR where $OPRQ$ is a parallelogram.

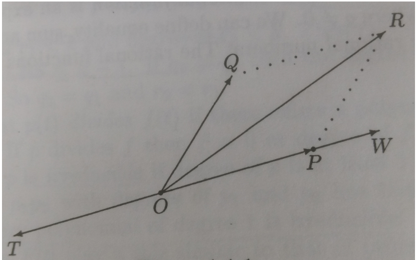


Figure 1.1.1

Certain parts of linear algebra are intimately related to geometry.

Also, if the strength of the force OP is doubled without changing the direction, the nrew fore is OS where S is the point on the lie OP such that $OS = 2OP$.

If the direction of the force OP is reversed without altering the magnitute, the new force is OT where T is the point on OP such that $OT = -OP$ with the usual convention.

Think Geometrically

One can probably make use of the vector space $\mathbb{R}^3$ by identifying a triple (x_1, x_2, x_3) of real numbers with the points in three dimensional Euclidean space

- to "think geometrically" about vector spaces ;
- to illustrate and motivate some of the ideas in linear algebra.

Most often we look at $\mathbb{R}, \mathbb{R}^2$ and $\mathbb{R}^3$ to understand the geometric meaning underlying the concepts.

By introducing a **coordinate system**, we can identity the plane of geometric vectors with $\mathbb{R}^2$; this was the great idea of Descartes's Analytic Geometry.

One way of getting new vector spaces from a given vector space V is to look at subsets S of V which form vector spaces by themselves. For example, the points of $\mathbb{R}^2$ lying on the x-axis themselves form a vector space and we call this a subspace of $\mathbb{R}^2$.

Definition 27.

Let V be a vector space over the field F. A **subspace** *of V is a subset W of V which is itself a vector space over F with the operations of vector addition and scalar multiplication on V.*

In other words, a subspace is a subset which is "**closed**" under additon and scalar mutiplication. Those operations follow the rules of the host space, without taking us outside the subspace.

A subspace W of a vector space V forms a vector space over the same field F under the operations: the restriction of "addition" to $W \times W$ and the restriction of "scalar multiplication" to $F \times W$. We denote the restricted operations by the same symbols.

Example 28.

The following subsets of the vector space $\mathbb{R}^n$ form non-trivial subspaces of $\mathbb{R}^n$:

(a) $\Big\{(x_1, \ldots, x_n) : x_1 = \cdots = x_m = 0 \text{ for any fixed } m,\ 1 \leq m < n\Big\}$;

(b) $\Big\{(x_1, \ldots, x_n) : x_1 + \cdots + x_n = 0\Big\}$;

(c) $\Big\{(x_1, x_2, x_3) : 2x_1 - 3x_2 + \sqrt{2}x_3 = 0, x_1 - 5x_3 = 0\Big\}$ *when* $n = 3$.

Exercise 29.

Fix $x_0 \in X$. Let $S = \{f : X \to \mathbb{R} | f(x_0) = 0\}$. Then S is a vector subspace of $F(X, \mathbb{R})$.

Example 30.

The following subsets of $\mathbb{R}$ form subspaces of $\mathbb{R}$ over $\mathbb{Q}$:

(a) $\mathbb{Q}$;

(b) $\left\{x + y\sqrt{2} + z\sqrt{3} : x, y, z \in \mathbb{Q}\right\}$.

Example 31.

If $P \subseteq S$, $\left\{f : f \in F^S \text{ and } f(x) = 0 \text{ for all } x \in P\right\}$ is a subspace of F^S. Also the set of all continuous functions and the set of all differentiable functions form subspaces of $\mathbb{R}^{\mathbb{R}}$.

Example 32.

(a) *If* $0 \leq m \leq n$, P_m *forms a subspace of* P_n. *Moreover, the subset of even polynomials as well as the subset of odd polynomials form subspaces of* P_n. *Note that* $\sum_{i=0}^{n-1} \alpha_i t^i$ *is even or odd according as* $\alpha_i = 0$ *whenever* i *is odd or even.*

(b) *Let* x, y *be two fixed vectors in a vector space* V *over* F. *Then* $W = \{\alpha x + \beta y : \alpha, \beta \in F\}$ *is a subspace of* V.

(c) *Consider the vector space, the power set of a set* Ω *over* $F = \{0, 1\}$ *with the operations defined earlier. For any nonempty subset* A *of* Ω, $\{\emptyset, A\}$ *is a subspace. For any distinct non-emtpy subsets* A *and* B *of* Ω, $\{\emptyset, A, B, A\Delta B\}$ *is another subspace.*

Note that a subspace W of a vector space is a vector space in its own right. **All nine rules of algebra do not need to be checked for a subset of a vector space over F to become a subspace of V because they are satisfied in the larger space and will automatically be satisfied in every subspace.**

Notice in particular that the **zero vector** will belong to every subspace.

Theorem 33.

A non-empty subset W of V is a subspace of V if and only if for each pair of vectors x, y in W and each scalar α in F the vector $\alpha x + y$ is again in W.

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1. The most extreme posibility for a subspace is to contain only one vector, the zero vector. It is a "zero-dimensional space," **containing only the zero vector**. This is **the smallest possible vector space**. Note that the empty set is not allowed.
   At the other extreme, **the largest subspace is the whole of the original space** - we can allow every vector into the subspace.
   If V is any vector space, V is a subspace of V ; the subset consisting of the zero vector alone is a subspace of V, called the **zero subspace** of V. Both are called **trivial subspaces**.
2. If the original space is $\mathbb{R}^3$, then the possible subspaces are easy to describe: $\mathbb{R}^3$ itself, any plane through the origin, any line through the origin, or the origin (the zero vector) alone.

Subspaces – Examples

3. If F^n, the set of n-tuples $(x_1, x_2, \ldots, x_n)$ with $x_1 = 0$ is a subspace ; however, the set of n-tuples with $x_1 = 1 + x_2$ is not a subspace $(n \geq 2)$.
4. The space of polynomial functions over the field F is a subspace of the space of all functions from F into F.
5. An $n \times n$ (square) matrix A over the field F is **symmetric** if $A_{ij} = A_{ji}$ for each i and j. The symmetric matrices form a subspace of the space of all $n \times n$ matrices over F.
6. An $n \times n$ (square) matrix A **over the field** $\mathbb{C}$ of complex numbers is **Hermitian** (or **self-adjoint**) if $A_{jK} = \overline{A_{kj}}$ for each j, k, the bar denoting complex conjugation.

Exercises 34.

Which of the following are subspaces of $\mathbb{R}^\infty$?

1. *All sequences like $(1, 0, 1, 0, \ldots)$ that include infinitely many rows.*
2. *All sequences $(x_1, x_2, x_3, \ldots)$ with $x_j = 0$ for some point onward.*
3. *All convergent sequences.*
4. *All geometric progression $(x_1, kx_1, k^2x_1, \ldots)$ allowing all k and x_1.*

Smallest Subspace Containing a Set

The distinction between a subset and a subspace is made clear by examples: Consider all vectors whose components are positive or zero. If the original space is the xy-plane $\mathbb{R}^2$, then this subset is the first quadrant; the coordinates satisfy $x \geq 0$ and $y \geq 0$. It is not a subspace, even though it contains zero and addition does leave us within the subset.

If $c = -1$ and $x = (1, 1)$, the multiple $cx = (-1, -1)$ is in the third quadrant instead of the first. If we include the third quadrant along with the first, then scalar multiplication is all right; every mutiple cx will staty in this subset, however the addition of $(1, 2)$ and $(-2, -1)$ gives a vector $(-1, 1)$ which is not in either quadrant.

The **smallest subspace** containing the first quadrant is the whole space $\mathbb{R}^2$.

If we start from the vector space of 3 by 3 matrices, then one possible subspace is **the set of lower triangular matrices**.

Another is **the set of symmetric matrices**. In both cases, the sums $A + B$ and the multiples cA inherit the properties of A and B. They are lower triangular if A and B are lower triangular, and they are symmetric if A and B are symmetric.

Of course, the zero matrix is in both **subspaces**.

Exercise 35.

What is the smallest subspace of 3×3 matrices that contains all symmetric matrices **and** *all lower triangular matrices? What is the largest subspace that is contained in* **both** *of those subspaces?*

Exercises 36.

1. *A real matrix* $A = (A_i j), 1 \leq i, j \leq n$ *is aaid to be* **symmetric** *if* $a_{ij} = a_{ji}$ *for all* $1 \leq i, j \leq n$. *Let* S_n *denote the set of all* $n \times n$ *symmetric real matrices. Then under the operations of matrix addition and scalar multiplicaition,* S_n *is a vector space.*
2. *A real matrix* $A = (A_i j), 1 \leq i, j \leq n$ *is aaid to be* **skew-symmetric** *if* $a_{ij} = -a_{ji}$ *for all* $1 \leq i, j \leq n$. *If* A_n *denotes the set of all skew-symmetric matrices, then* A_n *is a vector space under obvious addition and scalar multiplicaition. Note that both* S_n *and* A_n *are subsets of* $M(n, \mathbb{R})$.

Exercises 37.

1. *Write a general form of an* 2×2 *complex Hermitian matrix.*
2. *Check whether the set of all complex Hermitian matrices, a subspace of the space of all* $n \times n$ *matrices over* $\mathbb{C}$.
3. *Check whether the set of all* $n \times n$ *complex Hermitian matrices, a subspace of the space of all* $n \times n$ *matrices over* $\mathbb{C}$.
4. *Check whether the set of all* $n \times n$ *complex Hermitian matrices, a subspace of the space of all* $n \times n$ *matrices over* $\mathbb{R}$.

Exercises 38.

1. *Show that the polynomials (in one variable) of degress* ≤ 2, *of the form* $f(x) = a + bx + cx^2$, *are a subspace of* $F(\mathbb{R}, \mathbb{R})$.
2. *Show that the differentiable functions form a subspace of* $F(\mathbb{R}, \mathbb{R})$.

Exercises 39.

1. C^∞ *(the smooth functions, that is, we can differentiate as many times as we want) is a subspace of* $F(\mathbb{R}, \mathbb{R})$. *This subspace contains all polynomials in one variable, exponential functions,* $\sin x$ *and* $\cos x$, *for example.*
2. P *(the set of all polynomials in one variable) is a subspace of* $F(\mathbb{R}, \mathbb{R})$.
3. P_n *(the set of all polynomials in one variable of degree* $\leq n$*) is a subspace of* $F(\mathbb{R}, \mathbb{R})$.
4. *Show that the matrices* B *that commute with* $A = \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix}$ *form a subspace of* $\mathbb{R}^{2\times 2}$. *In general, show that the* $n \times n$ *matrices* B *that commute with any given* $n \times n$ *matrix* A *form a subspace of* $\mathbb{R}^{n\times n}$. *The degree of zero polynomial may be defined as* $-\infty$.
5. *Consider the set* W *of all non-invertible* 2×2 *matrices. Is* W *a subspace of* $\mathbb{R}^{2\times 2}$ *?*

Subspaces

Geometrically, think of the usual three-dimensional $\mathbb{R}^3$ and choose any plane through the origin. That plane is a vector space in its own right.

If we multiply a vector in the plane by 3, or -3, or any other scalar, we get a vector which lies in the same plane. If we add two vectors in the plane, their sum stays in the plane.

This plane illustrates one of the most fundamental ideas in the theory of linear algebra; it is a subspace of the original space $\mathbb{R}^3$.

Exercise 40.

The solution space of a system of homogeneous linear equations.

Let A be an $m \times n$ matrix over F. Prove that the set of all $n \times 1$ (column) matrices X over F such that $AX = 0$ is a subspace of the space of all $n \times 1$ matrices over F.

Lemma 41.

If A is an $m \times n$ matrix over F and B, C are $n \times p$ matrices over F, then

$$A(dB + C) = d(BC) + AC$$

for each scalar d in F.
Similarly, one can show that $(dB + C)A = d(BA) + CA$, if the matrix sums and products are defined. LA-1(P-2)L-2

Theorem 42.

Let V be a vector space over the field F. The intersection of any collection of subspaces of V is a subspace of V. LA-1(P-2)T-3

Exercise 43.

What about the union of any collection of subspaces of V ?

From Theorem (42) it follows that if S is any collection of vectors in V, then there is a smallest subspace of V which contains S. That is, the smallest subspace of V containing S is a subspace which contains S and which is contained in every other subspace containing S.

Subspace spanned by a set

Definition 44.

Let S be a set of vectors in a vector space V.
The **subspace** *W* **spanned** *by S is defined to be the intersection of all subspaces of V which contain S and it is denoted by $Sp(S)$. The set S is called a* **generating set** *of the subspace $W = Sp(S)$.*

When S is a finite set of vectors, $S = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$, we shall simply call W the **subspace spanned by the vectors** $\alpha_1, \alpha_2, \ldots, \alpha_n$.

Theorem 45.

The subspace spanned by a non-empty subset S of a vector space V is the set of all linear combinations of vectors in S. That is,

$$Sp(S) = \left\{ \sum_{i=1}^{k} \alpha_i x_i : x_i, \in S, \alpha_i \in F, 1 \leq i \leq k \text{ and } k \text{ is a postive integer} \right\}.$$

Note here that k, α_i, x_i are all arbitrarily chose from their respective domains. *LA-1(P-3)T-4*

Exercise 46.

For any subsets A and B of a vector space V, prove that

(a) A *is a subspace of* V *iff* $A = Sp(A)$;

(b) *If* $A \supseteq B$, *then* $Sp(A) \supseteq Sp(B)$;

(c) $Sp(Sp(A)) = Sp(A)$.

Exercise 47.

(a) *Say true or false : If* $A \subseteq B$ *and* $Sp(A) \supseteq B$, *then* $Sp(A) = Sp(B)$.

(b) *Can it happen that* $Sp(S') = Sp(S)$, *for subsets* $S' \subsetneq S \subseteq V$? *Illustrate with an example or an argument.*

Exercise 48.

Let v *and* $\{v_i\}_{i=1}^n$, *be vectors in a vector space* V.

Let $S' = \{v_i\}_{i=1}^n$ *and* $S = \{v\} \cup S'$.

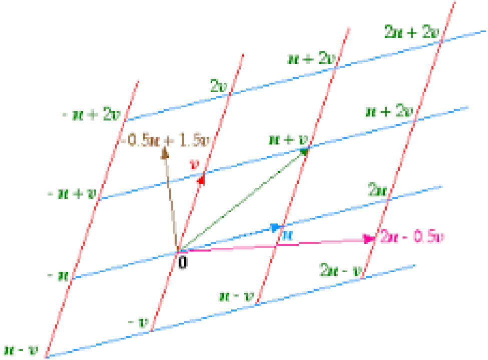
Then $L(S') = L(S)$ *if and only if there exist scalars* $\alpha_i \in \mathbb{R}, 1 \leq i \leq n$, *such that*

$$v = \sum_{i=1}^{n} \alpha_i v_i.$$

In particular, we see that $v \in L(\{v_1, \ldots, v_k\})$ *if and only if*

$$L(\{v_1, \ldots, v_k\}) = L(\{v, v_1, \ldots, v_k\}).$$

The following figure illustrates the geometric meaning of linear combination of u, v in $\mathbb{R}^2$. $Sp\{u, v\}$ creates a **mesh**.



Definition 49.

If $S_1, S_2, \ldots, S_k$ are subsets of a vector space V, the set of all sums

$$\alpha_1 + \alpha_2 + \cdots + \alpha_k$$

of vectors α_i in S_i is called the **sum** *of the subsets $S_1, S_2, \ldots, S_k$ and is denoted by*

$$S_1 + S_2 + \cdots + S_k$$

or by $\sum_{i=1}^{n} S_i$.

If $W_1, W_2, \ldots, W_k$ are subspaces of V, then the sum

$$W = W_1 + W_2 + \cdots + W_k$$

is easily seen to be a subspace of V which contains each of the subspaces W_i. From this it follows, that W is the subspace spanned by the union of $W_1, W_2, \ldots, W_k$.

Let V, W be vector spaces. Let us form the Cartesian product $V \times W$. Define addition and scalar multiplication on $V \times W$ as follows :

$$\begin{aligned} (v_1, w_1) + (v_2, w_2) &= (v_1 + v_2, w_1 + w_2) \\ \alpha(v, w) &= (\alpha v, \alpha w), \end{aligned}$$

where $(v_i, w_i) \in V \times W, \quad i = 1, 2, \quad \alpha \in \mathbb{R}, \quad (v, w) \in V \times W.$

Then $V \times W$ is a vector space. This vector space is usually denoted by $V \oplus W$ and called **direct sum** of V and W.

Exercise 50.

Extend the construction to define the directo sum $V_1 \oplus \cdots \oplus V_n$ of n vector spaces V_i, $1 \leq i \leq n$.

Example 51.

Let F be a subfield of the field $\mathbb{C}$ of complex numbers.

Suppose $\alpha_1 = (1, 2, 0, 3, 0), \alpha_2 = (0, 0, 1, 4, 0)$ *and* $\alpha_3 = (0, 0, 0, 0, 1)$.

The following are equivalent:

- *The subspace W of F^5 spanned by $\alpha_1, \alpha_2, \alpha_3$.*
- $W = \Big\{\alpha = (c_1, 2c_1, c_2, 3c_1 + 4c_2, c_3) : c_1, c_2, c_3 \in F\Big\}$.
- $W = \Big\{\alpha = (x_1, x_2, x_3, x_4, x_5) : x_2 = 2x_1, x_4 = 3x_1 + 4x_3\Big\}$.

Example 52.

Let F be a subfield of the field $\mathbb{C}$ of complex numbers and let V be the vector space of all 2×2 matrices over F.

Let $W_1 = \left\{ \begin{pmatrix} x & y \\ z & 0 \end{pmatrix} : x, y, z \in F \right\}$ *and* $W_2 = \left\{ \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} : x, y \in F \right\}$.

Then

- W_1 *and* W_2 *are subspaces of* V.
- $V = W_1 + W_2$.
- $W_1 \cap W_2 = \left\{ \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} : x \in F \right\}$.

We now see key examples of subspaces through matrices. They are tied directly to a $m \times n$ matrix A, and they give information about the system $Ax = b$.

The **column space** contains all linear combinations of the columns of A and it is denoted by $C(A)$. The system $Ax = b$ is solvable iff the vector b can be expressed as a combination of the columns of A. Then b is in the column space.

Example 53.

The matrices $A = \begin{pmatrix} 1 & 0 \\ 5 & 4 \\ 2 & 2 \end{pmatrix}$ *and* $B = \begin{pmatrix} 1 & 0 & 1 \\ 5 & 4 & 9 \\ 2 & 2 & 4 \end{pmatrix}$ *have the same column spaces.*
Note that the third column of B *is the sum of first and second columns of* B.

Example 54.

Let A be an $m \times n$ matrix over a field F. The **row vectors of** *A are the vectors in F^n given by*

$$\alpha_i = (A_{i1}, A_{i2}, \ldots, A_{in}), \ \ i = 1, 2, \ldots, m.$$

The subspace of F^n spanned by the row vectors of A is called the **row space** *of A.*

Nullspace : Another Example of a Subspace

The **nullspace of a matrix** consists of all vectors x such that $Ax = 0$ (i.e., the set of solutions to $Ax = 0$). It is denoted by $N(A)$.

- If $Ax = 0$ and $Ay = 0$, then $A(x + y) = 0$.
- If $Ax = 0$, then $A(cx) = 0$.

As both requirement are satisfied, $N(A)$ **is a subspace of** $\mathbb{R}^n$**.**

Note that both requirements fail if the right-hand side is not zero!

Example 55.

Let V be the space of all polynomial functions over F. Let S be the subset of V consisting of the polynomial functions $f_0, f_1, f_2, \ldots$ defined by

$$f_n(x) = x^n, \quad n = 0, 1, 2, \ldots.$$

Then V is the subspace spanned by the set S.

Applicaton of Linear Algebra in Finding Solution Space of Differential Equation

Consider the differential equation (DE)

$$f''(x) + f(x) = 0 \quad (or) \quad f'' = -f(x). \tag{1}$$

A solution (1) is a function $f(x)$ whose second derivative is the negative of the function itself.

For example, "$\sin x$" and "$\cos x$" are solutions of (1).

Applicaton of Linear Algebra in Finding Solution Space of Differential Equation

Can we find any other solutions ?

Note that the solution set of this DE is closed under addition and under scalar multiplication.

It follows that all "linear combinations"

$$f(x) = c_1 \sin x + c_2 \cos x \tag{2}$$

The following exercise shows that every solution of 1 is of the form 2. Thus the functions "$\sin x$" and "$\cos x$" span the solution space V of the DE $f''(x) = -f(x)$.

Exercise 56.

(a) *Show that if* $g(x)$ *is in* V*, then the function* $(g(x))^2 + (g'(x))^2$ *is constant. [Hint : Consider the derivative.]*

(b) *Show that if* $g(x)$ *is in* V *with* $g(0) = f'(0) = 0$*, then* $g(x) = 0$ *for all* x*.*

(c) *If* $f(x)$ *is in* V*, then* $g(x) = f(x) - f(0)\cos x - f'(0)\sin x$ *is in* V *as well. Verify that* $g(0) = 0$ *and* $g'(0) = 0$*.*

We can conclude that $g(x) = 0$ *for all* x*, so that*

$$f(x) = f(0)\ \sin x + f'(0)\ \cos x.$$

It follows that the functions "$\sin x$*" and "*$\cos x$*" span* V*, as claimed.*

Solution Space of Differential Equation

Since the solution set V of the DE (1) is closed under addition and scalar multiplication, we can say that V is a "subspace" of $F(\mathbb{R}, \mathbb{R})$.

Even there are infinitely many solutions, using the language of linear algebra, we can conclude that the functions "$\sin x$" and "$\cos x$" form a "basis" of the "solution space" V, so that the "dimension" of our DE is 2.

Thus, the solutions of our DE form a two-dimensional subspace of $F(\mathbb{R}, \mathbb{R})$ with basis "$\sin x$" and "$\cos x$".

Exercises 57.

(a) *Which of the following set of vectors* $\alpha = (a_1, a_2, \ldots, a_n)$ *in* $\mathbb{R}^n$ *are subspaces of* $\mathbb{R}^n$ $(n \geq 3)$ *?*

(i) *all* α *such that* $a_1 \geq 0$ *;*
(ii) *all* α *such that* $a_1 + 3a_2 = a_3$ *;*
(iii) *all* α *such that* $a_2 = a_1^2$ *;*
(iv) *all* α *such that* $a_1 a_2 = 0$ *;*
(v) *all* α *such that* a_2 *is rational.*

(b) *Let* V *be the (real) vector space of all functions* f *from* $\mathbb{R}$ *to* $\mathbb{R}$*. Which of the following sets of functions are subspaces of* V*?*

(i) *all* f *such that* $f(x^2) = f(x)^2$ *;*
(ii) *all* f *such that* $f(0) = f(1)$ *;*
(iii) *all* f *such that* $f(3) = 1 + f(-5)$ *;*
(iv) *all* f *such that* $f(-1) = 0$ *;*
(v) *all* f *which are continuous.*

Exercises 58.

(a) *Is the vector* $(3, -1, 0, -1)$ *in the subspace of* $\mathbb{R}^5$ *spanned by the vectors* $(2, -1, 3, 2), (-1, 1, 1, -3)$, *and* $(1, 1, 9, -5)$ *?*

(b) *Let* W *be the set of all* $(x_1, x_2, x_3, x_4, x_5)$ *in* $\mathbb{R}^5$ *which satisfy*

$$\begin{array}{rcrcrcrcrcl} 2x_1 & - & x_2 & + & \frac{4}{3}x_3 & - & x_4 & & & = & 0 \\ x_1 & & & + & \frac{2}{3}x_3 & & & - & x_5 & = & 0 \\ 9x_1 & - & 3x_2 & + & 6x_3 & - & 3x_4 & - & 3x_5 & = & 0. \end{array}$$

Find a finite set of vectors which spans W.

Exercises 59.

(a) *Let F be a field and let n be a positive integer ($n \geq 2$). Let V be the vector space of all $n \times n$ matrices over F. Which of the following sets of matrices A in V are subspaces of V?*

(i) *all invertible A ;*
(ii) *all non-invertible A ;*
(iii) *all A such that $AB = BA$, where B is some fixed matrix in V ;*
(iv) *all A such that $A^2 = A$.*

(b) (i) *Prove that the only subspaces of $\mathbb{R}^1$ are $\mathbb{R}^1$ and the zero subspace.*
(ii) *Prove that a subspace of $\mathbb{R}^2$ is $\mathbb{R}^2$, or the zero subspace, or consists of all scalar multiples of some fixed vector in $\mathbb{R}^2$. (The last type of subspace is, intuitively a straight line through the origin.)*
(iii) *Can you describe the subspaces of $\mathbb{R}^3$?*

Exercises 60.

(a) *Let W_1 and W_2 be subspaces of a vector space V such that the set-theoretic union of W_1 and W_2 is also a subspace. Prove that one of the spaces W_i is contained in the other.*

(b) *Let V be the vector space of all functions from $\mathbb{R}$ into $\mathbb{R}$; let V_e be the subset of even functions, $f(-x) = f(x)$; let V_o be the subset of odd functions $f(-x) = -f(x)$.*

   (i) *Prove that V_e and V_o are subspaces of V.*

   (ii) *Prove that $V_e + V_o = V$.*

   (iii) *Prove that $V_e \cap V_o = \{0\}$.*

(c) *Let W_1 and W_2 be subspaces of a vector space V such that $W_! + W_2 = V$ and $W_1 \cap W_2 = \{0\}$. Prove that for each vector α in V there are* **unique** *vectors α_1 in W_1 and α_2 in W_2 such that $\alpha = \alpha_1 + \alpha_2$.*

Exercises 61.

(a) *Let* $X := \{*\}$ *be a singleton set and let* V *be a vector space. Let* $W = \{*\} \times V$. *We can turn* W *into a vector space as follows:*

$$\begin{aligned} (*, v_1) + (*, v_2) &= (*, v_1 + v_2) \\ \alpha(*, v) &= (*, \alpha v), \end{aligned}$$

where $v_1, v_2 \in V, \quad \alpha \in \mathbb{R}, \quad v \in V$.

(b) $V := \mathbb{Q}$. *On* $\mathbb{Q}$ *we have a natural addition, namely, the addition of rationl numbers. However, if* $\alpha \in \mathbb{R}$ *is irrational and* $r \in \mathbb{Q}$ *then* $\alpha r \in \mathbb{R}$ *but not in* $\mathbb{Q}$. *Then* V *is not a vector space over* $\mathbb{R}$.

Exercises 62.

Let $P_i, 1 \leq i \leq n$ be continuous functions on $[a, b] \subseteq \mathbb{R}$. Let V be the set of n-times continuously differentiable solutions f on $[a, b]$ of a linear differential equation

$$y^{(n)} + P_1(x)y^{(n-1)} + \cdots + P_n(x)y = 0.$$

The set V of solutions of the differential equation is a vector space with pointwise operations.

Additional Exercises

Students are encouraged to go through the exercises given in the following textbooks.

Text Book	Pages
"Linear Algebra with Applications" by Otto Bretscher	162,163
"Linear Algebra" by A. Ramachandra Rao and P. Bhimasankaram	21, 22 28,29,30

References

- Kenneth Hoffman and Ray Kunze, "*Linear Algebra*", Second Edition, Prentice-Hall of India, 1990 (pages mainly from 28 to 40).
- Otto Bretscher, "*Linear Algebra with Applications*", Third Edition, Pearson, 2007 (pages mainly from 152 to 163).
- A. Ramachandra Rao and P. Bhimasankaram, "*Linear Algebra*", Hindustan Book Agency, 2000 (pages mainly from 14 to 30).
- S. Kumaresan, "*Linear Algebra - A Geometric Approach*", PHI Learning Pvt. Ltd., 2011 (pages mainly from 13 to 25).