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Computational Linear Algebra - MA 703
Problem Sheet 4

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1. True or false? If V, W are vector spaces and $T : V \to W$ is a linear map and $\{v_1, \ldots, v_n\}$ is a linearly independent set of vectors in V, then $\{T(v_i)\}_{i=1}^n$ is linearly independent.

2. Can you construct a linear map $T : \mathbb{R}^2 \to \mathbb{R}^4$ such that $\texttt{Im}(T) = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 + x_2 + x_3 + x_4 = 0\}$?

3. Can you construct a linear map $T : \mathbb{R}^2 \to \mathbb{R}^3$ such that $\texttt{Im}(T) = \{(x, y, z) \in \mathbb{R}^3 : x + y + z = 0\}$?

4. Think of a matrix A, as a linear map which takes the jth element of the standard basis of $\mathbb{R}^n$ to the jth column C_j. How the column space is nothing other than $\texttt{Im}(A)$? Explain.

5. Let V be the vector space of real 2×2 matrices and let $C = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}$ Define $T : V \to V$ by $T(A) = CA - 2A^T$ where A^T is the transpose of A. Is T linear?

6. Give two transformations $T : \mathbb{R}^3 \to \mathbb{R}^3$ whose transformation matrix is same in any choice of basis (same basis on both domain and codomain).

7. Find the *range* and *kernel* of $T : \mathbb{R}^3 \to \mathbb{R}^2$ define by $T(x, y, z) = (x, y)$.

8. Write the projection matrix which projects $(1, 2, 3)$ on to the line passing through $(1, 1, 1)$ and $(0, 0, 0)$.

9. Consider the mapping T from $\mathbb{R}^3$ to $\mathbb{R}^3$ that rearranges the components in increasing order. For example, $T(3, 2, 6) = (2, 3, 6)$. Is this a linear mapping? With explanation, say yes or no.

10. Find suitable bases on $\mathbb{R}^3$ and $\mathbb{R}^3$ so that a matrix of the linear map $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by
$$T(x, y, z) = (2x + y, 2y + z, x + y + z)$$
assumes a lower triangular matrix.

11. Write the matrix of the linear map
$$T(x, y, z) = (2x + 3y - z, x - y)$$
with respect to the bases $\{(1, 1, 1), (1, 0, 0), (1, 1, 0)\}$ in domain and $\{(0, 1), (1, 1)\}$ in codomain spaces in the respective orders.

12. If $T : V \to W$ is an injective (one-to-one) linear map between real vector spaces V and W and if $v_1, v_2, \ldots, v_n$ are linearly independent in V, then show that $Tv_1, Tv_2, \ldots, Tv_n$ are linearly independent in W.

13. Which of the following map is linear? In each case, $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

    (a) $Tx = \begin{bmatrix} x_1 - x_2 \\ x_1 / x_2 \end{bmatrix}$

    (b) $Tx = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

    (c) $Tx = \begin{bmatrix} 7x_1 - 4x_2 \\ x_1 x_2 \end{bmatrix}$

    (d) $Tx = \begin{bmatrix} x_1 + 11x_2 \\ x_1 + 11x_2 \end{bmatrix}$

14. The range of the linear map from $\mathbb{R}^2$ to $\mathbb{R}^2$ given by the matrix $A = \begin{bmatrix} 2 & 3 \\ -6 & -9 \end{bmatrix}$

(a) All of $\mathbb{R}^2$.

(b) A line through the origin having slope -3.

(c) A line through the points $(2,3)$ and $(-6,-9)$.

(d) A line through 0 having slope $2/3$.

(e) None of these.

15. Which of the following is true? $T(w,x,y,z) = (w+x+y+z, x+y+z, z)$

(a) Nullity $(T) \leq$ Rank (T)

(b) Nullity $(T) \geq$ Rank (T)

(c) Nullity $(T) =$ Rank (T)

(d) None of the above.

16. Which of the following is a linear map?

(a) $T(1,0,0) = (0,0,0)$,
$T(0,0,1) = (0,0,0)$ and
$T(0,0,1) = (0,0,0)$.

(b) $T(1,1,1) = (0,0,1)$,
$T(0,1,1) = (0,1,1)$ and
$T(0,0,1) = (1,1,1)$.

(c) $T(1,1,-1) = (1,1,1)$,
$T(-1,1,1) = (1,1,1)$ and
$T(1,-1,1) = (1,1,-1)$.

(d) $T(-1,-1,2) = (1,1,1)$,
$T(-1,2,-1) = (2,2,2)$ and
$T(2,-2,-1) = (3,3,3)$.

17. Which of the following is true?

(a) T and $-T$ have same ranks.

(b) T and $2T$ have same ranks.

(c) T and T^2 have same ranks.

(d) T and T^* have same ranks.

18. Let T be a linear map between two different linear spaces V and W. Which of the following are subspaces of V?

(a) $C(T)$ and $N(T)$.

(b) $C(T^*)$ and $N(T)$.

(c) $N(T)$ and $N(T^*)$.

(d) $C(T)$ and $N(T^*)$.

19. Find the matrix for the linear map that projects each point in $\mathbb{R}^3$ perpendicularly onto the xy-plane and then rotates it in that plane through an angle of $45°$.

20. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation which transforms the points $(1,1,-1)$, $(1,-1,0)$ and $(-1,0,0)$ into the points $(-1,0,0)$, $(1,-1,0)$ and $(1,1,-1)$ respectively. Find $T^{100}(2,-3,4)$ and $T^{99}(-1,2,3)$.

21. Let $T : V \to W$ be linear. Let V^* and W^* be their duals. We define a map $T^* : W^* \to V^*$ as follows. Given $g \in W^*$, $Tg \in V^*$ is given by $T^*g(v) := g(Tv)$ for all $v \in V$. Show that T^* is a linear map. It is called the *adjoint* of T.