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Computational Linear Algebra - MA 703
Problem Sheet 5

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1. Check whether the following are inner products or not.
   (a) $\langle z, w\rangle := \mathtt{Re}(z\overline{w})$ on $\mathbb{C}$.
   (b) $\langle (x_1, x_2), (y_1, y_2)\rangle := y_1(2x_1 + x_2) + y_2(x_1 + x_2)$ on $\mathbb{R}^2$.
   (c) $\langle A, B\rangle := \mathtt{tr}(AB^t)$ on $\mathbb{M}(n, \mathbb{R})$. The trace of the matrix A, $\mathtt{tr}(A)$ is the sum of diagonals.
2. Prove that two nonzero vectors orthogonal to each other are linearly independent.
3. Say true or false. Zero is the only vector, orthogonal to every vector of V.
4. Let $W_i, i = 1, 2,$ be vector subspaces of V. Assume that each vector W_1 is orthogonal to W_2 and vice-versa. Then prove that $W_1 \cap W_2 = \{0\}$.
5. Let $x = (\alpha, \beta, \gamma)$ be a nonzero vector in $\mathbb{R}^3$.
   (a) Find a basis of $W := x^{\perp}$.
   (b) Give a pair of equations whose solution set is the line joining the origin and v.
6. Give an example of a 2×2 matrix whose columns are orthogonal but not the rows.
7. Apply the Gram-Schmidt process to the vectors $v_1 = (1, 0, 3), v_2 = (2, 2, 0)$ and $v_3 = (3, 1, 2)$ to find an orthonormal basis for $\mathbb{R}^3$.
8. Find the values of a for which the planes $ax - y + z = 1$ and $3ax + ay - 2z = 5$ are perpendicular.
9. Find the norm and angle between the vectors $(2, 3, -4, 5)$ and $(-5, 6, 7, 2)$. State Cauchy-Schwartz inequality.
10. Apply the Gram-Schmidt algorithm to the columns of the matrix $\begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 1 & 2 & 3 \end{pmatrix}$. Use the order in which they occur in the matrix.
11. Let $x_1, x_2, \ldots, x_k$ form an orthonormal set.
   (a) Show that $\| \sum_{i=1}^k \alpha_i x_i \|^2 = \sum_{i=1}^k \|\alpha_i\|^2$.
   (b) If z is the residual of x on $\{x_1, x_2, \ldots, x_k\}$, show that
   $$\|z\|^2 = \|x\|^2 - \| \sum_{i=1}^k \langle x, x_i\rangle x_i \|^2 = \|x\|^2 - \sum_{i=1}^k |\langle x, x_i\rangle|^2.$$
   (c) **Bessel's inequality:**
   $$\|x\|^2 \geq \sum_{i=1}^k |\langle x, x_i\rangle|^2$$
   for any x. Show also that equality holds iff $x \in Sp(\{x_1, x_2, \ldots, x_k\})$.

12. Show that $\langle x, y \rangle = 0$ for all y iff $x = 0$.

13. Let $B = \{x_1, x_2, \ldots, x_k\}$ be an orthonormal set in a finite-dimensional inner product space V. Show that the following statements are equivalent:

    (a) B is maximal,

    (b) $\langle x, x_i \rangle = 0$ for $i = 1, 2, \ldots, k \Rightarrow x = 0$,

    (c) B generates V,

    (d) if $x \in V$ then $x = \sum_{i=1}^{k} \langle x, x_i \rangle x_i$,

    (e) if $x, y \in V$ then $\langle x, y \rangle = \sum_{i=1}^{k} \langle x, x_i \rangle . \langle x_i, y \rangle$,

    (f) if $x \in V$ then $\|x\|^2 = \sum_{i=1}^{k} |\langle x, x_i \rangle|^2$.

14. Let x, y, u and v belong to $\mathbb{R}^n$. Then show that

$$\langle x + iy, u + iv \rangle := u^T x + v^T y$$

    is an inner product on the vector space $\mathbb{C}^n$ over $\mathbb{R}$. What is its connection with the canonical inner product on $\mathbb{C}^n$?

15. Prove that every orthogonal set of nonzero vectors in an inner product space is linearly independent. Say true or false, in $\mathbb{R}^n$, there does not exist an orthogonal set of $n + 1$ nonzero vectors.

16. Prove that an orthogonal set of n nonzero vectors in $\mathbb{R}^n$ is a basis for $\mathbb{R}^n$.

17. In $\mathbb{R}^n$, use the standard inner product. What is the orthogonal projection of the vector $x = (1, 2, 3)$ on the vector $y = (3, 2, 1)$?

18. Is the vector $x = (3, 2, 1)$ in the orthogonal complement of the pair of vectors $\{u, v\}$, where $v = (5, -2, -1)$ and $v = (2, -3, 15)$?

19. Let A be a subset of an inner product space V. Prove that the orthogonal complement of A is the same as the orthogonal complement of $span(A)$.

20. Explain why the hypotheses $\|x + y\| = \|y\|$ and $x \neq 0$ lead to $\langle x, y \rangle < 0$.