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Computational Linear Algebra - MA 703
Problem Sheet 8

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1. Suppose $A \in \mathbb{R}^{n\times n}$ and $x \in \mathbb{R}^r$ are given. Give a saxpy algorithm for computing the first column of $M = (A - x_1 I)\cdots(A - x_r I)$.
2. In the conventional 2×2 matrix multiplication $C = AB$, there are eight multiplication :
$$a_{11}b_{11}, a_{11}b_{12}, a_{21}b_{11}, a_{21}b_{12}, a_{12}b_{21}, a_{12}b_{22}, a_{22}b_{21} \text{ and } a_{22}b_{22}.$$
Make a table that indicates the order that these multiplications are performed for the ijk, jik, kij, ikj, jki and kji matrix multiply algorithms.
3. Give an algorithm for computing $C = (xy^T)^k$ where x and y are n-vectors.
4. Specify an algorithm for computing $(XY^T)^k$ where $X, Y \in \mathbb{R}^{n\times 2}$.
5. Formulate an outer product algorithm for the update $C = AB^T + C$ where $A \in \mathbb{R}^{m\times r}, B \in \mathbb{R}^{n\times r}$, and $C \in \mathbb{R}^{m\times n}$.
6. Suppose we have real $n \times n$ matrices C, D, E and F. Show how to compute real $n \times n$ matrices A and B with just three real $n \times n$ matrix multiplications so that $(A + iB) = (C + iD)(E + iF)$.
(Hint: Compute $W = (C + D)(E - F)$.)
7. Give an algorithm that overwrites A with A^2 where $A \in \mathbb{R}^{n\times n}$ is
   (a) upper triangular and
   (b) square.

   Strive for a minimum workspace in each case.
8. Suppose $A \in \mathbb{R}^{n\times n}$ is upper Hessenberg and that scalars $\lambda_1, \ldots, \lambda_r$ are given. Give a saxpy algorithm for computing the first column of $M = (A - \lambda_1 I)\cdots(A - \lambda_r I)$.
9. Give a column saxpy algorithm for the $n \times n$ matrix multiplication problem $C = AB$ where A is upper triangular and B is lower triangular.
10. Extend "**Band Gaxpy**" algorithm so that it can handle rectangular band matrices. Be sure to describe the underlying data structure.
11. $A \in \mathbb{R}^{n\times n}$ is **Hermitian** if $A^H = A$. If $A = B + iC$, then it is easy to show that $B^T = B$ and $C^T = -C$. Suppose we represent A in an array $A.herm$ with the property that $A.herm(i, j)$ houses b_{ij} if $i \geq j$ and c_{ij} if $i > j$. Using the data structure write a matrix-vector multiply function that computes $\text{Re}(z)$ and $\text{Im}(z)$ from $\text{Re}(x)$ and $\text{Im}(x)$ so that $z = Ax$.
12. Suppose $X \in \mathbb{R}^{n\times p}$ and $A \in \mathbb{R}^{n\times n}$, with A symmetric and stored by diagonal. Give an algorithm that computes $Y = X^T AX$ and stores the result by diagonal. Use separate arrays for A and Y.
13. Suppose $a \in \mathbb{R}^n$ is given and that $A \in \mathbb{R}^{n\times n}$ has the property that $a_{ij} = a_{|i-j|+1}$. Give an algorithm that overwrites y with $Ax + y$ where $x, y \in \mathbb{R}^n$ are given.

14. Suppose $a \in \mathbb{R}^n$ is given and that $A \in \mathbb{R}^{n\times n}$ has the property that $a_{ij} = a_{((i+j-1) \mod n)+1}$. Give an algorithm that overwrites y with $Ax + y$ where $x, y \in \mathbb{R}^n$ are given.

15. Develop a compact store-by-diagonal scheme for unsymmetric band matrices and write the corresponding gaxpy algorithm.

16. Suppose p and q are n-vectors and that $A = (a_{ij})$ is defined by $a_{ij} = a_{ji} = p_i q_j$ for $1 \leq i \leq j \leq n$. How many flops are required to compute $y = Ax$ where $x \in \mathbb{R}^n$ is given?

17. Prove that if

$$A = \begin{bmatrix} A_{11} & \cdots & A_{1r} \\ \vdots & \ddots & \vdots \\ A_{q1} & \cdots & A_{qr} \end{bmatrix}$$

is a blocking of the matrix A, then

$$A^T = \begin{bmatrix} A_{11}^T & \cdots & A_{q1}^T \\ \vdots & \ddots & \vdots \\ A_{1r}^T & \cdots & A_{qr}^T \end{bmatrix}.$$

18. Suppose n is even and define the following function from $\mathbb{R}^n$ and $\mathbb{R}$:

$$f(x) = x(1:2:n)^T x(2:n) = \sum_{i=1}^{n/2} x_{2i-1} x_{2i}$$

(a) Show that if $x, y \in \mathbb{R}^n$ then

$$x^T y = \sum_{i=1}^{n/2} (x_{2i-1} + y_{2i})(x_{2i} + y_{2i-1}) - f(x) - f(y)$$

(b) Now consider the $n \times n$ matrix multiplication $C = AB$. Give an algorithm for computing this product that requires $n^3/2$ multiplies once f is applied to the rows of A and the columns of B.

19. Consider the matrix product $D = ABC$ where $A \in \mathbb{R}^{m\times r}, B \in \mathbb{R}^{r\times n}$ and $C \in \mathbb{R}^{n\times q}$. Assume that all the matrices are strored by column and that the time required to execute a unit-stride saxpy operation of length k is of the form $t(k) = (L + k)\mu$ where L is a constant and μ is the cycle time. Based on this model, when is it more economical to compute D as $D = (AB)C$ instead of as $D = A(BC)$? Assume that all matrix multiple are done using the *jki*, (gaxpy) algorithm.

20. What is the total time spent in *jki* variant on the saxpy operations assuming that all the matrices are stored by column and that the time required to execute a unit-stride saxpy operation of length k is of the form $t(k) = (L + k)\mu$ where L is a constant and μ is the cycle time? Specialize the algorithm so that it efficiently handles the case when A and B are $n \times n$ and upper triangular. Does it follow that the triangular implementation is six times fastes as the flop count suggets?

21. Give an algorithm for computing $C = A^T BA$ where A and B are $n \times n$ and B is symmetric. Arrays should be accessed in unit stride fashion within all innermost loops.