Ramanujan's Method

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Srinivasa Ramanujan

Srinivasa Ramanujan (22 December 1887 - 26 April 1920) was an Indian mathematician who, with almost no formal training in pure mathematics, made extraordinary contributions to mathematical analysis, number theory, infinite series, and continued fractions.

G.H. Hardy

J.E. Littlewood

In 1913, the English mathematician **G. H. Hardy** received a strange letter from an unknown clerk in Madras, India.

The ten-page letter contained about 120 statements of theorems on infinite series, improper integrals, continued fractions, and number theory.

Hardy and his collaborator J. E. Littlewood understood that the results "must be true because, if they were not true, no one would have had the imagination to invent them". Thus was Srinivasa Ramanujan introduced to the mathematical world.

Ramanujan's Method

During his short life (1887-1920), Ramanujan independently compiled nearly 3900 results (mostly identities and equations).

Nearly all his claims have now been proven correct, although a small number of these results were actually false and some were already known.

He stated results that were both original and highly unconventional, such as the **Ramanujan prime** and the **Ramanujan theta function**, and these have inspired a vast amount of further research.

We dicuss an iterative procedure to determine the **smallest** root of the equation

$$f(x) = 0$$

where $f(x)$ is of the form

$$f(x) = 1 - (a_1x + a_2x^2 + a_3x^3 + \cdots).$$

Procedure

To explain the method of procedure, we consider the quadratic equation

$$f(x) = a_0x^2 + a_1x + a_2 = 0,$$

with the roots x_1 and x_2, such that $|x_1| < |x_2|$.

Then the equation defined by

$$\begin{aligned} \phi(x) &= a_2x^2 + a_1x + a_0 = 0 \\ &\Rightarrow 1 + \frac{a_1}{a_0}x + \frac{a_2}{a_0}x^2 = 0 \end{aligned}$$

will have roots $\frac{1}{x_1}$ and $\frac{1}{x_2}$ such that $\frac{1}{|x_1|} > \frac{1}{|x_2|}$.

Now,

$$\frac{1}{\phi(x)} = \left(1 + \frac{a_1}{a_0}x + \frac{a_2}{a_0}x^2\right)^{-1}$$

can be written as

$$\begin{aligned}
\left(1 + \frac{a_1}{a_0}x + \frac{a_2}{a_0}x^2\right)^{-1} &= \frac{k_1}{x - \frac{1}{x_1}} + \frac{k_2}{x - \frac{1}{x_2}} \\
&= \frac{-k_1 x_1}{1 - x x_1} + \frac{-k_2 x_2}{1 - x x_2} \\
&= -k_1 x_1 \left(1 - x x_1\right)^{-1} - k_2 x_2 \left(1 - x x_2\right)^{-1} \\
&= \sum_{i=0}^{\infty} b_i x^i, \qquad \text{where } b_i = -\sum_{r=1}^{2} k_r x_r^{i+1}
\end{aligned}$$

Then,

$$\frac{b_{i-1}}{b_i} = \frac{k_1 x_1^i + k_2 x_2^i}{k_1 x_1^{i+1} + k_2 x_2^{i+1}} = \frac{\frac{k_1}{k_2}\left(\frac{x_1}{x_2}\right)^i + 1}{\frac{k_1}{k_2}\left(\frac{x_1}{x_2}\right)^{i+1} + 1} \cdot \frac{1}{x_2}$$

Since $\frac{x_1}{x_2} < 1$, it follows that

$$\lim_{i \to \infty} \frac{b_{i-1}}{b_i} = \frac{1}{x_2},$$

which is the smallest root. This is the basis of Ramanujan's method which is outlined below.

How to apply the method?

To find the smallest root of $f(x) = 0$, we consider $f(x)$ in the form

$$f(x) = 1 - (a_1x + a_2x^2 + a_3x^3 + \cdots),$$

and then write

$$\left[1 - (a_1x + a_2x^2 + a_3x^3 + \cdots)\right]^{-1} = b_1 + b_2x + b_3x^2 + \cdots$$

which implies that

$$1 + (a_1x + a_2x^2 + a_3x^3 + \cdots) + (a_1x + a_2x^2 + a_3x^3 + \cdots)^2 + \cdots$$

$$= b_1 + b_2x + b_3x^2 + \cdots$$

To find b_i, we equate coefficients of like powers of x on both sides of the above equation. We get the recurrence relation

$$\begin{aligned} b_1 &= 1 \\ b_k &= a_{k-1}b_1 + a_{k-2}b_2 + \cdots + a_1 b_{k-1}, \quad k \geq 2. \end{aligned}$$

It is known that the ratios

$$\left(\frac{b_i}{b_{i+1}}\right)_{i=1}^{\infty}$$

approach, to the **smallest** root of $f(x) = 0$.

The **first five approximations** are

$$\frac{b_1}{b_2}, \frac{b_2}{b_3}, \frac{b_3}{b_4}, \frac{b_4}{b_5}, \text{ and } \frac{b_5}{b_6}.$$

In order to calcuate the first five approximations, we have to calculate the values $b_1, b_2, \ldots, b_6$.

Exercises

Find the root of the follwoing equations by Ramanujan's method.

1. $xe^x = 1$
2. $\sin x = 1 - x$

Note :

Ramanujan's method is preferable when the given function consists of an infinite series.

Reference

- S.S. Sastry, *Introductory Methods of Numerical Analysis*, Fourth Edition, Prentice-Hall, India.