Linear Independence, Basis and Dimension

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Overview

We usually associate “dimension” with something geometrical.

We turn now to the task of finding a suitable algebraic definition of the **dimension** of a vector space.

This will be done through the concept of a basis for the space.

We discuss linear independence, basis and dimension in two lectures.

Suppose we are given a generating set A for a subspace S. One question that arises naturally is: is there any redundancy in A in the sense that some proper subset of A generates S? To answer this, we start with

Definition 1.

Let V be a vector space over F. A subset S of V is said to be **linearly dependent** *(or simply,* **dependent***) if there exist distinct vectors $x_1, x_2, \dots, x_n$ in S and scalars $\alpha_1, \alpha_2, \dots, \alpha_n$ in F, not all of which are 0, such that*

$$\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n = 0.$$

A set which is not linearly dependent is called **linearly independent***. If the set S contains only finitely many vectors $x_1, x_2, \dots, x_n$, we sometimes say that $x_1, x_2, \dots, x_n$ are dependent (or independent) instead of saying S is dependent (or independent).*

An infinite set S is said to be **linearly independent** *if each finite subset of it is linearly independent.*

The following are easy consequences of the definition.

- Any set which contains a linearly dependent set is linearly dependent.
- Any subset of a linear independent set is linearly independent.
- Any set which contains the 0 vector is linearly dependent.
- A set S of vectors is linearly independent if and only if each finite subset of S is linearly independent, i.e., if and only if for distinct vectors $x_1, x_2, \ldots, x_n$ of S,

$$\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n = 0$$

  implies each $\alpha_i = 0$.

If $x_1, x_2, \ldots, x_n$ are linearly independent, then $x_{i_1}, x_{i_2}, \ldots, x_{i_n}$ are also linearly independent, for any permutation $(i_1, i_2, \ldots, i_n)$ of $(1, 2, \ldots, n)$, so the linear indepedence of a set of vectors is well-defined.

Exercise 2.

Give an example for the fact that "not all zero" cannot be replaced by "all non-zero" in the definition of linear independence.

Exercise 3.

Let $X = \{a_1, a_2, \ldots\}$ where $a_1, a_2, \ldots$ form a sequence of distinct real numbers. Consider the function f_i belonging to the vector space $\mathbb{R}^X$ taking value 1 at $x = a_i$ and 0 elsewhere. Then prove that the infinite set $\{f_1, f_2, \ldots\}$ is linearly independent.

Example 4.

In the vector space $\mathbb{R}$ *over the field* $\mathbb{Q}$*, the sets* $\{1, \sqrt{2}\}$ *and* $\{\sqrt{2}, \sqrt{3}\}$ *are linearly independent and the set* $\{\sqrt{2}, \sqrt{3}, \sqrt{12}\}$ *is linearly independent.*

Example 5.

The vectors $1, t, t^2, \ldots, t^{n-1}$ *in the vector space* P_n *form a linearly independent set.*

The vectors $1, t, t^2, \ldots$ *in the vector space* P *form an infinite linearly independent set.*

Linear Independency Test

To prove that a set

$$\{x_1, x_2, \dots, x_n\}$$

is linearly independent, we should claim that the zero vector can only be expressed as zero (scalar) linear combination of x_i's, $1 \leq i \leq n$.

That is, we claim that the equation

$$\alpha_1 x_1 + \cdots + \alpha_n x_n = 0$$

has only the trivial solution $\alpha_1 = \cdots = \alpha_n = 0$.

Suppose $\alpha_1 x_1 + \cdots + \alpha_n x_n = 0$ for some scalars $\alpha_1, \dots, \alpha_n$.

We should claim that each α_i is zero, to prove $\{x_1, x_2, \dots, x_n\}$ is linearly independent.

Linear Dependency Test

To prove a set $\{x_1, x_2, \ldots, x_n\}$ is linearly dependent, we should find scalars $\alpha_1, \ldots, \alpha_n$ not all zero such that

$$\alpha_1 x_1 + \cdots + \alpha_n x_n = 0.$$

That is, the zero vector can be expressed as some non-zero (scalar) linear combination of x_i's $1 \leq i \leq n$.

Definition 6.

Let V be a vector space. A **basis** *for V is a linearly independent set of vectors in V which spans the space V.*

The space is **finite dimensional** *if it has a finite basis.*

Example 7.

Let F be a field and in F^n let S be the subset consisting of the vectors $e_1, e_2, \dots, e_n$ defined by

$$\begin{aligned} e_1 &= (1, 0, 0, \dots, 0) \\ e_2 &= (0, 1, 0, \dots, 0) \\ &\vdots \\ e_n &= (0, 0, 0, \dots, 1). \end{aligned}$$

The set $S = \{e_1, e_2, \dots, e_n\}$ is a basis for F^n, called the **standard basis** *of F^n.*

Basis of the Row Space of A

Let A be an $m \times n$ matrix and let R be a row reduced echelon matrix which is row-equivalent to A. **For an echelon matrix like R, the row space is clear. It contains all combinations of the rows, as every row space does.**

To every echelon matirx, with r pivots and r nonzero rows: Its nonzero rows are independent, and its row space has dimension r. Fortunately, it is equally easy to deal with the original matrix A.

The row space of A has the same dimension r as the row space of R, and it has the same bases, because the two row spaces are the same. The reason is that each elementary operation leaves the row space unchanged.

The rows in R are combinations of the original rows in A. It is true that A and R have different rows, but the combinations of the rows are identical. It is those combinations that make up the row space.

Basis of $F^{n\times 1}$

Exercise 8.

Let P be an invertible $n \times n$ matrices with entries in the field. Prove that the columns of P form a basis for the space of column matrices, $F^{n\times 1}$.

Basis for the Solution Space

Let A be an $m \times n$ matrix and let S be the solution space for the homogeneous system $AX = 0$. The purpose of elimination is to simplify a system of linear equations without changing any of the solutions.

Let R be a row reduced echelon matrix which is row-equivalent to A. Then the system $AX = 0$ is reduced to $Rx = 0$, and this process is reversible. **Therefore S is also the solution space for the system $RX = 0$.**

A basis for the solution space can be constructed by reducing to $RX = 0$, which has $n - r$ free variables – corresponding to the columns of R that do not contain pivots. Then, in turn, we give to each free variable the value 1, to the other free variables the value 0, and solve $RX = 0$ by back substitution for the remaining (basic) variables. The $n - r$ vectors produced in this way are a basis for the solution space for $AX = 0$.

Refer Linear Algebra by Kenneth Hoffman and Ray Kunze, page 42, for details.

An Example of an Infinite Basis

Exercise 9.

Let F be a subfield of the complex numbers and let V be the space of polynomial functions over F. Note that the members of V are the functions from F into F which have a rule of the form

$$f(x) = c_0 + c_1x + \cdots + c_nx^n.$$

Let $f_k(x) = x^k, \ k = 0, 1, 2, \ldots$.

Prove that the infinite set $\{f_0, f_1, f_2, \ldots\}$ *is a basis for V. [Hints :*

- *A function f from F into F is a polynomial function iff there exists an integer n and scalars $c_0, c_1, \ldots, c_n$ such that* $f = c_0f_0 + c_1f_1 + \cdots + c_nf_n$.
- *A polynomial of degree n with complex coefficients cannot have more than n distinct roots.]*

Refer Linear Algebra by Kenneth Hoffman and Ray Kunze, page 43, for details.

An infinite basis has been found for the space V of polynomial functions over F. But V might have a finite basis as it is not immediate from the definition. We shall see in the next theorem that this possibility doesn't occur.

Example 10.

The space of polynomial functions over F cannot have a finite basis.

Refer Linear Algebra by Kenneth Hoffman and Ray Kunze, page 43, for details.

Caution :

Infinite bases have nothing to do with "infinite linear combinations".

"Linear combination" always refers to "finite linear combination."

Theorem 11.

Let V be a vector space which is spanned by a finite set of vectors $\beta_1, \beta_2, \ldots, \beta_m$. Then any independent set of vectors in V is finite and contains no more than m elements. LA-1(P-4)T-5

To prove the above theorem, we need the following facts:

- Equivalent systems of linear equations have exactly the same equations. LA-1(P-4)L-1
- If A and B are row-equivalent $m \times n$ matrices, the homogeneous systems of linear equations $AX = 0$ and $BX = 0$ have exactly the same solutions. LA-1(P-4)L-2
- If A is an $m \times n$ matrix and $m < n$ (fat matrix), then the homogeneous system of linear equations $AX = 0$ has a non-trivial solution. LA-1(P-5)L-3

Corollary 12.

If V if finite-dimensional vector space, then any two bases of V have the same (finite) number of elements. LA-1(P-7)C-6

Definition 13.

The above corollary allows us to define the **dimension** *of a finite-dimensional vector space as the number of elements in a basis for V, and it is denoted by "dim V."*

A vector space V is called **finite dimensional** *if it has finite basis. Otherwise, the space is called* **infinite dimensional**.

The space P of all polynomials is inifinte dimensional (as was known to Peano in 1888).

Corollary 14.

Let V be a finite-dimensional vector space and let $n = \dim V$. Then

1. *any subset of V which contains more than n vectors is linearly dependent (every basis is a maximal linearly independent set);*
2. *no subset of V which contains less than n vectors can span V (every basis is a minimal spanning set).*

Define maximal and minimal

Exercises 15.

(a) *If F is a field, find the dimensions of the space F^n and the matrix space $F^{m\times n}$.*

(b) *If A is an $m \times n$ matrix, find a basis and the dimension of the solution space for A.*

(c) *If A is an $m \times n$ matrix, find a basis and the dimension of the column space for A.*

Basis of the zero subspace

If V is any vector space over F, the zero subspace of V is spanned by the vector 0, but $\{0\}$ is a linearly dependent set and **not a basis**.

For this reason, we shall agree that **the zero subspace has dimension** 0.

The empty set spans $\{0\}$, because the intersection of all subspaces containing the empty set is $\{0\}$, and the empty set is linearly independent (convention), becasue it contains no vectors (the linear independency condition is vacuously true).

Hence **the emtpy set is a basis for the zero subspace.**

Lemma 16.

Let S be a linearly independent subset of a vector space V. Suppose β is a vector in V which is not in the subspace spanned by S. Then the set obtained by adjoining β to S is linearly independent. LA-1(P-8)L-7

Theorem 17.

If W is a subspace of a finite-dimensional vector space V, every linearly independent subset of W is finite and is part of a (finite) basis for W. LA-1(P-8)T-7

Corollary 18.

If W is a proper subspace of a finite-dimensional vector space V, then W is finite-dimensional and $\dim W < \dim V$. LA-1(P-9)C-9

Corollary 19.

In a finite-dimensional vector space V, every non-empty linearly independent set of vectors is part of a basis.

Corollary 20.

Let A be an $n \times n$ matrix over a field F, and suppose that row vectors of A form a linearly independent set of vectors in F^n. Then A is invertible.
LA-1(P-10)T-10

Here we recall matrix multiplication :

- **Matrix Multiplication - Column wise.** Each column of AB is the product of a matrix and a column: column j of $AB = A$ times column j of B.
- **Matrix Multiplication - Row wise.** Each row of AB is the product of a row and a matirx: row i of $AB =$ row i of A times B.

Theorem 21.

If W_1 and W_2 are finite-dimensional subspaces of a vector space V, then $W_1 + W_2$ is finite-dimensional and

$$dim\ W_1 + dim\ W_2 = dim\ (W_1 \cap W_2) + dim\ (W_1 + W_2).$$

LA-1(P-10)T-11

Linear Independence for Finite Sequences

A remark about linear independence and dependence

We discussed these concepts for sets for vectors $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$. Usually, the set $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is presumed that no two of the vectors are identical bcause "**a set is a well-defined collection of distinct objects.**"

We now define "linear independence" for finite sequences (**ordered** *n*-**tuples**) of vectors : $\alpha_1, \alpha_2, \ldots, \alpha_n$.

Linear Independence for Finite Sequences

What is the difference between a finite sequence $\alpha_1, \alpha_2, \ldots, \alpha_n$ and a set $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$?

There are two differences **identity and order**.

Identity : As not like a set, in a sequence $\alpha_1, \alpha_2, \ldots, \alpha_n$ all the α_i's may be the same vector.

Thus, if $\alpha_1, \alpha_2, \ldots, \alpha_n$ are linearly independent, they are distinct.

The dimension of a finite-dimensional vector space V **is the largest** n such that some n-tuple of vectors in V is linearly independent – and so on.

Order :

The elements of a sequence are enumerated in a specific order. A set is a collection of objects with no specified arrangement or order. Of course, to describe the set we may list its members, and that requires choosing an order.

But, the order is not part of the set. The sets $\{1, 2, 3, 4\}$ and $\{4, 3, 2, 1\}$ are identical, whereas $1, 2, 3, 4$ is quite a different sequence from $4, 3, 2, 1$. The order aspect of sequences has no bearing on questions of independence, dependence, etc., because dependence (as defined) is not affected by the order.

The sequence $\alpha_n, \ldots, \alpha_1$ is dependent if and only if the sequence $\alpha_1, \ldots, \alpha_n$ is dependent. We shall discuss later how order is important.

Exercises 22.

(a) *Prove that if two vectors are linearly dependent, one of them is a scalar multiple of the other.*

(b) *Are the vectors* $\alpha_1 = (1, 1, 2, 4), \alpha_2 = (2, -1, -5, 2)$, $\alpha_3 = (1, -1, -4, 0)$ *and* $\alpha_4 = (2, 1, 1, 6)$ *linearly independent in* $\mathbb{R}^4$*?*

(c) *Find a basis for the subspace of* $\mathbb{R}^4$ *spanned by the above four vectors.*

(d) *Show that the vectors* $\alpha_1 = (1, 0, -1), \alpha_2 = (1, 2, 1), \alpha_3 = (0, -3, 2)$ *form a basis for* $\mathbb{R}^3$*. Express each of the standard basis vectors as linear combinations of* α_1, α_2*, and* α_3*.*

(e) *Find three vectors in* $\mathbb{R}^3$ *which are linearly dependent, and are such that any two of them are linearly independent.*

(f) *Let* V *be the vector space of all* 2×2 *matrices over the field* F*. Prove that* V *has dimension 4 by exhibiting a basis for* V *which has four elements.*

Exercises 23.

(a) *Let V be the vector space of all 2×2 matrices over the field F. Let W_1 be the set of matrices of the form*

$$\begin{pmatrix} x & -x \\ y & z \end{pmatrix}$$

and let W_2 be the set of matrices of the form

$$\begin{pmatrix} a & b \\ -a & c \end{pmatrix}.$$

(i) *Prove that W_1 and W_2 are subspaces of V.*

(ii) *Find the dimension of $W_1, W_2, W_1 + W_2, W_1 \cap W_2$.*

Exercises 24.

(a) *Let* V *be the vector space of all* 2×2 *matrices over the field* F. *Find a basis* $\{A_1, A_2, A_3, A_4\}$ *for* V *such that* $A_i^2 = A_i$ *for each* i.

(b) *Let* V *be a vector space over a subfield* F *of the complex numbers. Suppose* α, β *and* γ *are linearly independent vectors in* V. *Prove that* $(\alpha + \beta), (\beta + \gamma)$, *and* $(\gamma + \alpha)$ *are linearly independent.*

(c) *Let* V *be the vector space over the field* F. *Suppose there are a finite number of vectors* $\alpha_1, \alpha_2, \ldots, \alpha_r$ *in* V *which span* V. *Prove that* V *is finite-dimensional.*

Exercises 25.

(a) *Let V be the set of all 2×2 matrices A with complex entries which satisfy $A_{11} + A_{22} = 0$.*

  (i) *Show that V is a vector space over the field of real numbers, with the usual operations of matrix addition and multiplication of a matrix by a scalar.*

  (ii) *Find a basis for this vector space.*

  (iii) *Let W be the set of all matrices A in V such $A_{21} = -\overline{A}_{12}$ (the bar denotes complex conjugation). Prove that W is a subspace of V and find a basis for W.*

(b) *Prove that the space of all $m \times n$ matrices over the field F has dimension mn, by exhibiting a basis for this space.*

(c) *Let V be the set of real numbers. Regard V as a vector space over the field of rational numbers, with the usual operations. Prove that this vector space is not finite-dimensional.*

PART - 2

For any vectors $u_1, u_2, \ldots, u_n$ we have that $0u_1 + 0u_2 + \cdots + 0u_n = 0$. This is called the **trivial representation** of 0 as a linear combination of $u_1, u_2, \ldots, u_n$.

This motivates a definition of "**linear dependence**". For a set to be linearly dependent, there must exist a non-trivial representation of 0 as a linear combination of vectors in the set.

Definition 26.

A subset S of a vector space V is called **linearly dependent** *if there exist a finite number of distinct vectors $v_1, v_2, \ldots, v_n$ in S and scalars $a_1, a_2, \ldots, a_n$,* **not all zero**, *such that*

$$a_1v_1 + a_2v_2 + \cdots + a_nv_n = 0.$$

Note that the zero on the right is the **zero vector**, not the number zero.

Linear Dependent Set

- Any set containing the zero vector is linearly dependent.
- If $m > n$, a set of m vectors in $\mathbb{R}^n$ is dependent.

A subset S of a vector space V is then said to be **linearly independent** if it is not linearly dependent.

In other words, **a set is linearly independent if the only representations of** 0 **as a linear combination of its vectors are trivial representations**.

Linear Dependent Set

More generally, let V be a vector space over $\mathbb{R}$, and let $\{v_i : i \in I\}$ be a family of elements of V. The family is **linearly dependent** over $\mathbb{R}$ if there exists a family $\{a_j : j \in J\}$ of elements of $\mathbb{R}$, not all zero, such that $\sum_{j \in J} a_j v_j = 0$, where the index set J is a nonempty, finite subset of I.

A set $\{v_i : i \in I\}$ of elements of V is **linearly independent** if the corresponding family $\{v_i : i \in I\}$ is not linearly dependent.

Exercise 27.

A family is dependent if a member is in the linear span of the rest of the family.

Spanning a Subspace

Definition 28.

A set of vectors S **spans** *a subspace W if $W = \langle S \rangle$; that is, if every element of W is a linear combination of elements of S.*

In other words, we call the **subspace** W **spanned by a set** S if all possible linear combinations produce the space W.

If S spans a vector space V (we denote $\mathrm{Sp}(S) = V$), then every set containing S is also a spanning set of V.

Linear Independent Set

Definition 29.

A set B of vectors in a vector space V is said to be a **basis** *if B is linearly independent and spans V.*

From the definition of a basis B, every element of V can be written as linear combination of elements of B, **in one and only way**.

Definition 30.

The number of elements of a basis B of a vector space V is called the **dimension** *of V.*

Examples 31.

1. *The coordinate vectors $e_1, e_2, \dots, e_n$ coming from the identity matrix spans $\mathbb{R}^n$. Hence the dimension of $\mathbb{R}^n$ is n*
2. *The vector space $\mathcal{P}(x)$ of all polynomials in x over $\mathbb{R}$ has the (infinite) subset $1, x, x^2, \dots$ as a basis, so $\mathcal{P}(x)$ has infinite dimension.*

Linear Independent Set

In a subspace of dimension k, no set of more than k vectors can be independent, and no set of fewer than k vectors can span the space.

- Any linearly independent set in V can be extended to a basis, by adding more vectors if necessary.
- Any spanning set in V can be reduced to a basis, by discarding vectors if necessary.

Hence basis is a **maximal linearly independent set**, or a **minimal spanning set**.

Four Fundamental Subspaces

Let A be an $m \times n$ matrix of real entries.

1. The **column space** of A, $\mathcal{C}(A)$ is the space spanned by columns of A. That is,
$$\mathcal{C}(A) := \{Ax : x \in \mathbb{R}^n\}.$$
2. The **null space** of A, $\mathcal{N}(A)$ is the solution set of the matrix equation $Ax = 0$. That is,
$$\mathcal{N}(A) = \{x \in \mathbb{R}^n : Ax = 0\}.$$
3. The **row space** of A, is the space spanned by rows of A. It is same as the column space of A^T. That is,
$$\mathcal{R}(A^T) := \{y^T A : y \in \mathbb{R}^m\}.$$
4. The **left nullspace** of A is the nullspace of A^T. That is,
$$\mathcal{N}(A^T) := \{y \in \mathbb{R}^m : y^T A = 0\}.$$

Echelon Form

A matrix that has undergone Gaussian elimination is said to be **in row echelon form** or, more properly, "**reduced echelon form**" or "**row-reduced echelon form**". Such a matrix has the following characteristics :

1. All **zero rows** are **at the bottom** of the matrix.
2. The **leading entry of each nonzero row** after the **first occurs** to the **right of the leading entry of the previous row**.
3. The **leading entry** in any nonzero row is 1.
4. All entries in the column **above and below a leading** 1 **are zero**.

The Row Space

- Use Gaussian elimination to transform $[A|b]$ into echelon form $[U|c]$. Transforming to the echelon matrix means that we are taking linear combinations of the rows of a matrix A to come up with the matrix U.
- We could reverse the process and get back to A, by row operations again. Therefore, the row space of A equals the row space of U.
- If 2 matrices are row equivalent, then their row spaces are the same.
- The nonzero rows (rewritten as column vectors) of the matrix U form a **BASIS** for the row space.

Example for a basis of the row space of A

$$U = \begin{pmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ is the echelon matrix of } A = \begin{pmatrix} 1 & 2 & 1 & 2 \\ 1 & 2 & 1 & 3 \\ 3 & 6 & 3 & 7 \end{pmatrix}.$$

A basis for the row space of A is the set of non-zero rows of U (rewritten as column vectors). In our example, a basis for the row space is

$$\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Notice the number of elements in the basis, which is the number of non-zero rows in U, is just the number of pivots.

The Column Space

- The column space of a matrix consists of **ALL** linear combinations of the columns of the matrix.
- Reduce A to the echelon matrix U, by row operations.
- Find the pivot variables $x_{i_1}, x_{i_2}, \ldots, x_{i_r}$ where $r = \text{rank}(A)$ is the total number of pivots. A basis for the column space of A is the set of columns $i_1, i_2, \ldots, i_r$ in the original matrix A.
- That is, the columns of the original matrix corresponding to those columns containing **PIVOTS** form a **BASIS** for the column space of the matrix.
- Notice the number of elements in the basis, which is the number of non-zero rows in U, is just the number of pivots. This means **the column space and the row space of a matrix always have the same dimension**.

Example for a basis of the column space of A

$$U = \begin{pmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ is the echelon matrix of } A = \begin{pmatrix} 1 & 2 & 1 & 2 \\ 1 & 2 & 1 & 3 \\ 3 & 6 & 3 & 7 \end{pmatrix}.$$

Since the pivot variables of A are x_1 and x_4, a basis for the column space is

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 7 \end{pmatrix} \right\}.$$

Caution. We had to choose columns from the original matrix A, because Gaussian elimination changes the column picture at each step.

To see this in our example, just note that the columns of U all have a zero in the third entry: if these spanned the column space of A then every column of A would also have to have zero in the third entry–which is false!

Nullspace

- The system $Ax = 0$ is reduced to $Ux = 0$, where U is an echelon matrix, and this process is reversible.
- The nullspace of A is the same as the nullspace of U.
- Find the free variables $x_{j_1}, x_{j_2}, \ldots, x_{j_{n-r}}$ where $n - r$ is the number of columns of A without a pivot. There is a basis for the nullspace of A, with one vector associated to each free variable. Taking each free variable one at a time, set that free variable as 1 and the other free variables as 0. Then solve $Ux = 0$ for the vector x with this choice and put x in the basis. Repeat for each free variable.
- The $n - r$ "special solutions" to $Ux = 0$ provides a **BASIS** for the nullspace.

Example for a basis of the null space of A

$U = \begin{pmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ is the echelon matrix of $A = \begin{pmatrix} 1 & 2 & 1 & 2 \\ 1 & 2 & 1 & 3 \\ 3 & 6 & 3 & 7 \end{pmatrix}$.

The free variables are x_2 and x_3. Taking $x_2 = 1, x_3 = 0$ and $x_2 = 0, x_3 = 1$ and solve $Ux = 0$ for each choice. A basis for the nullspace of is

$$\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

Left Nullspace

- Use Gaussian elimination to transform $[A|b]$ into echelon form $[U|c]$.
- A basis for the left nullspace of A has $m - r$ vectors, which is the number of zero rows in U.
- For each zero row, put a vector in the basis whose entries are the coefficients of the vector b in the entry of c corresponding to the zero row.

Example for a basis of the left null space of A

$$[A \mid b] = \begin{bmatrix} 1 & 2 & 1 & 2 & b_1 \\ 1 & 2 & 1 & 3 & b_2 \\ 3 & 6 & 3 & 7 & b_3 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 & 2 & b_1 \\ 0 & 0 & 0 & 1 & b_2 - b_1 \\ 0 & 0 & 0 & 0 & b_3 - 2b_1 - b_2 \end{bmatrix} = [U \mid c].$$

There is one zero row of U. As an equation the row represents $0 = -2b_1 - b_2 + b_3$ (note we listed the b_is in order). Thus a basis for the left nullspace is

$$\left\{ \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix} \right\}.$$

In general, the number of elements in this basis will equal the number of zero rows.

Dimensions of Four Fundamental Subspaces

Let A be $m \times n$ with rank r. Using the bases above, we observe the following:

- dim $R(A) = \dim R(A^T) = r$ (the number of pivots).
- dim $N(A) = n - r$ (the number of free columns).
- dim $N(A^T) = m - r$ (the number of zero rows).

Notice that the column space and the row space of a matrix have the same dimension (even though the vectors in each subspace live in a different ambient space, $\mathbb{R}^m$ and $\mathbb{R}^n$.)

The nullspace and row space live in $\mathbb{R}^n$; the left nullspace and column space live in $\mathbb{R}^m$.

Fundamental Theorem of Linear Algebra (Part I)

There is an important relationship between the dimensions of the subspaces in each of these pairs of subspaces, which is shown by the following theorem.

Theorem 32 (Fundamental Theorem of Linear Algebra (Part I)).

Let A be $m \times n$ with rank r. Then

- $dim\ R(A^T) + dim\ N(A) = n$.
- $dim\ R(A) + dim\ N(A^T) = m$.

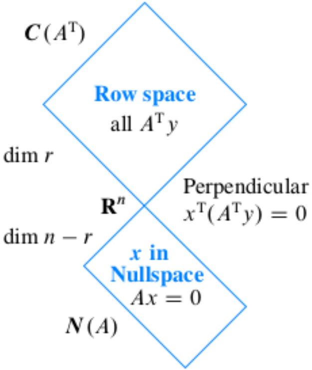
We call the dimension of $N(A)$, **nullity** of A.

The first equation is also called the **rank-nullity law, or, rank-nullity theorem** because rank $A = \dim\ R(A^T)$.

To Temember Their Dimensions !

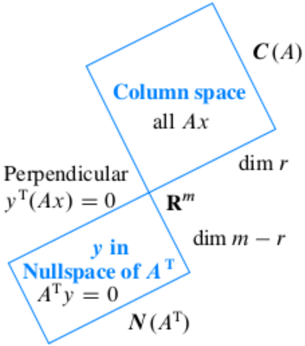
To remember what are the dimensions of the four fundamental subspaces, it is best just to think about where the bases for each subspace comes from :

- The bases for the column space and row space come from the pivots, so the dimension of each of these subspaces is the rank of the matrix.
- The basis for the nullspace comes from the free columns, so the dimension of the nullspace is the number of free columns.
- The basis for the left nullspace is obtained from the zero rows, so the dimension of the left nullspace is the number of zero rows.



$C(A^T)$
Row space
all $A^T y$
dim r
Perpendicular
$x^T(A^T y) = 0$
$\mathbf{R}^n$
dim $n - r$
x in
Nullspace
$Ax = 0$
$N(A)$

Figure : Left Nullspace and Column Space in $\mathbb{R}^m$



Additional Exercises

Students are encouraged to go through the exercises given in the following textbooks.

Text Book	Pages
"Linear Algebra with Applications" by Otto Bretscher	162,163
"Linear Algebra" by A. Ramachandra Rao and P. Bhimasankaram	21, 22 28,29,30

References

- Kenneth Hoffman and Ray Kunze, "*Linear Algebra*", Second Edition, Prentice-Hall of India, 1990 (pages mainly from 40 to 49).
- Otto Bretscher, "*Linear Algebra with Applications*", Third Edition, Pearson, 2007 (pages mainly from 152 to 163).
- A. Ramachandra Rao and P. Bhimasankaram, "*Linear Algebra*", Hindustan Book Agency, 2000.
- S. Kumaresan, "*Linear Algebra - A Geometric Approach*", PHI Learning Pvt. Ltd., 2011.