Department of Mathematical and Computational Sciences, NITK, Surathkal
Linear Algebra and Matrices (MA 904)
https://sam.nitk.ac.in/sites/default/MA904-ps-6.pdf

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Problem Sheet 6

1. Think of a matrix A, as a linear map which takes the jth elements of the standard basis of $\mathbb{R}^n$ to the jth column C_j. How the column space is nothing other than $\texttt{Im}(A)$? Explain.

2. Let $x = (\alpha, \beta, \gamma)$ be a nonzero vector in $\mathbb{R}^3$.
   (a) Find a basis of $W := x^{\perp}$.
   (b) Give a pair of equations whose solution set is the line joining the origin and v.

3. Show that a vector space V over F has a unique basis iff either '$d(V) = 0$' or '$d(V) = 1$ and $|F| = 2$'.

4. Prove or disprove: If A, B and C are pair-wise disjoint subsets of V such that $A \cup B$ and $A \cup C$ are bases of V, then $\texttt{Sp}(B) = \texttt{Sp}(C)$.

5. Let $x_1, x_2, \ldots, x_n$ be fixed distinct real numbers.
   (a) Show that $\ell_1(t), \ell_2(t), \ldots, \ell_n(t)$ form a basis of $\mathcal{P}_n$, where $\ell_i(t) = \Pi_{i \neq j}(t - x_j)$. This basis leads to what is known as *Lagrange's interpolation formula.* If $f(t) \in \mathcal{P}_n$ is written as $\sum_{i=1}^{n} \alpha_i \ell_i(t)$, show that $\alpha_i = f(x_i)/\ell_i(x_i)$.
   (b) Show that $\psi_1(t), \psi_2(t), \ldots, \psi_n(t)$ form a basis of $\mathcal{P}_n$, where $\psi_i(t) = 1$ and $\psi_i(t) = \Pi_{j=1}^{i=1}(t - x_j)$ for $i = 2, \ldots, n$. This basis leads to what is known as *Newton's divided difference formula.*

6. Extend $A = \{(1, 1, \ldots, 1)\}$ to a basis of $\mathbb{R}^n$.

7. Let S and T be subspaces of a vector space V with $d(S) = 2, d(T) = 3$ and $d(V) = 5$. Find the minimum and maximum possible values of $d(S + T)$ and show that every (integer) value between these can be attained.

8. Show that the distributive law
$$S \cup (T + W) = (S \cup T) + (S \cup W)$$
is false for subspaces. However prove that it holds whenever $S \supseteq T$ or $S \supseteq W$. This latter result is known as the *modular law.*

9. The sum of two subspaces S and T is said to be *direct* (or S and T *independent*) if any vector in $S + T$ can be expressed in a unique way as $x + y$ with $x \in S$ and $y \in T$. Prove that the following statements are equivalent.
   (a) $S + T$ is direct.
   (b) $S \cup T = \{0\}$.
   (c) If $x \in S - \{0\}$ and $y \in T - \{0\}$, then x, y are linearly independent.
   (d) $0 = x + y, x \in S, y \in T \Rightarrow x = 0$ and $y = 0$.
   (e) $d(S + T) = d(S) + d(T)$.

10. Say true or false: A complement of a subspace is unique.