Topology Induced by Families of Functions

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Overview

We are given with the following :

- a non-empty set X
- a family $\mathcal{F}$ of functions such that each f in $\mathcal{F}$ maps X into a topological space (Y_f, τ_f).

If we wish to find a topology for X that makes every member of $\mathcal{F}$ continuous, then the discrete topology on X makes every function defined on X continuous (whether $\mathcal{F}$ has many or few functions.)

That topology does not depend on the family $\mathcal{F}$ of functions and it may have too many open sets. So there are too many open covers and not all of them have finite subcovers.

In the discrete topology, only finite sets are compact and the eventually constant sequences are the only convergent sequences.

Aim

We are given with the following :

- a non-empty set X
- a family $\mathcal{F}$ of functions such that each f in $\mathcal{F}$ maps X into a topological space (Y_f, τ_f).

Our aim is to find a smallest topolgy (denoted by $\tau_{\mathcal{F}}$) for X with respect to which each member of $\mathcal{F}$ is continuous.

Existence of smallest topology

As the following result shows, existence of smallest topology is always possible.

Theorem 1.

Let X be a set and let $\mathfrak{F}$ be a family of functions and $\{(Y_f, \mathfrak{T}_f) : f \in \mathfrak{F}\}$ a family of topological spaces such that each f in $\mathfrak{F}$ maps X into the corresponding Y_f. Then there is a smallest topology for X with respect to which each member of $\mathfrak{F}$ is continuous. That is, there is a unique topology $\mathfrak{T}_{\mathfrak{F}}$ for X such that

(1) *each f in $\mathfrak{F}$ is $\mathfrak{T}_{\mathfrak{F}}$-continuous; and*

(2) *if $\mathfrak{T}$ is any topology for X such that each f in $\mathfrak{F}$ is $\mathfrak{T}$-continuous, then $\mathfrak{T}_{\mathfrak{F}} \subseteq \mathfrak{T}$.*

The topology $\mathfrak{T}_{\mathfrak{F}}$ has $\{f^{-1}(U) : f \in \mathfrak{F}, U \in \mathfrak{T}_f\}$ as a subbasis.

Topologizing family of functions for X

- We say that every member of $\mathcal{F}$ is τ-continuous, when τ is a topology on X makes each member of $\mathcal{F}$ continuous.
- $\mathcal{F}$ is called a **topologizing family of functions for** X.
- The topology $\tau_{\mathcal{F}}$ is the $\mathcal{F}$-**topology of** X or **the topology of** X **induced by** $\mathcal{F}$**.**
- The collection

$$\left\{ f^{-1}(U) : f \in \mathcal{F}, U \in \tau_f \right\}$$

  is the **standard subbasis** for the topology.
- The **standard basis** for the topology is the collection of all sets that are intersection of finitely many members of this subbasis.

Product Topology

Given the following :

- $\Big\{(X_\alpha, \tau_\alpha) : \alpha \in \mathbb{I}\Big\}$, a family of topological spaces.
- $X = \prod_{\alpha\in\mathbb{I}} X_\alpha$. That is, $X = \Big\{x : \mathbb{I} \to \cup X_\alpha \;\text{ and }\; x^{(\alpha)} \in X_\alpha\Big\}$. By the Axiom of Choice, $X \neq \emptyset$. We denote a member x of X as $(x^{(\beta)})_{\beta\in\mathbb{I}}$.
- For each α in $\mathbb{I}$, $\pi_\alpha : X \to X_\alpha$ defined by

$$\pi_\alpha(x) = \pi_\alpha((x^{(\beta)})_{\beta\in\mathbb{I}}) = x^{(\alpha)}.$$

  π_α is called the **projection** from X to X_α.
- $\mathcal{F} = \{\pi_\alpha : \alpha \in \mathbb{I}\}$.

The **product topology** of X is the $\mathcal{F}$-topology of X. The **topological product** of the family of topological spaces is the cartesian product with the product topology.

Subbasis for topological product

Fix $\alpha_0 \in \mathbb{I}$ and an open subset U of X_{α_0},

$$\pi_{\alpha_0}^{-1}(U) = \prod_{\alpha \in \mathbb{I}} U_\alpha$$

where $U_{\alpha_0} = U$ and $U_\alpha = X_\alpha$ when $\alpha \neq \alpha_0$. The collection

$$\left\{ \pi_\alpha^{-1}(U) : \alpha \in \mathbb{I}, U \in \tau_\alpha \right\}$$

is the standard subbasis for the product topology $\tau_{\mathcal{F}}$ of X.

What about the convergence of nets in the topological product?

Recall

Theorem 2.

Suppose that $\mathfrak{S}$ *is a subbasis for the topology of a topological space* X, *that* $(x_\alpha)_{\alpha \in I}$ *is a net in* X, *and that* $x \in X$. *Then* $x_\alpha \to x$ *if and only if the following is true : For every member* U *of* $\mathfrak{S}$ *that contains* x, *there is an* α_U *in* I *such that* $x_\alpha \in U$ *whenver* $\alpha_U \preceq \alpha$.

Net convergence in a topological product

Theorem 3.

Let $\left\{X^{(\alpha)} : \alpha \in \mathbb{I}\right\}$ *be a family of topological spaces and let* X *be their topological product. Suppose that* $(x_\beta)_{\beta \in J}$ *is a net in* X *(here* J *is the index set for the net) and* x *is a member of* X*. The following are equivalent :*

1. $x_\beta \to x \quad \left(\quad x_\beta = \{(x_\beta^{(\alpha)})_{\alpha \in \mathbb{I}}\}_{\beta \in J}\ \text{and}\ \ x = (x^{(\alpha)}) \quad \right)$
2. $x_\beta^{(\alpha)} \to x^{(\alpha)}$ *for each* α *in* $\mathbb{I}$
3. $\pi_\alpha(x_\beta) \to \pi_\alpha(x)$ *for each* α *in* $\mathbb{I}$
4. $f(x_\beta) \to f(x)$ *for each* f *in* $\mathcal{F}$.

Net convergence in a topological product is equivalent to coordinate convergence.

What about the convergence in $\mathcal{F}$-topology ?

The above result generalizes to arbitrary $\mathcal{F}$-topologies.

Theorem 4.

Let X be a set and $\mathcal{F}$ a topologizing family of functions for X. Suppose that (x_α) is a net in X and x is a member of X. Then

$$x_\alpha \to x$$

with respect to the $\mathcal{F}$-topology iff

$$f(x_\alpha) \to f(x)$$

for each f in $\mathcal{F}$.

Net convergence in $\mathcal{F}$-topology is equivalent to coordinate convergence.

A consequence of coordinate convergence

Theorem 5.

Let W be a topological space and let X ba a set topologized by a family $\mathcal{F}$ of functions and let g be a function from W into X. Then g is continuous iff $f \circ g$ is continuous for each f in $\mathcal{F}$.

Relation between $\mathcal{F}$-topology and product topology

Let X be a set and let $\mathcal{F}$ be a family of functions such that each f in $\mathcal{F}$ maps X into a topological space (Y_f, τ_f). We have two topologies:

- X with $\mathcal{F}$-topology
- $\prod_{f \in \mathcal{F}} Y_f$ with the product topology.

Define the map

$$x \mapsto (f(x))_{f \in \mathcal{F}} \tag{1}$$

from X with $\mathcal{F}$-topology into $\prod_{f \in \mathcal{F}} Y_f$ with the product topology.

Is the map given by (1) a homeomorphism?

Definition 6.

Let X be a set and let $\mathcal{F}$ be a family of functions each of which has domain X. $\mathcal{F}$ is **separating** *or* **total** *if, for each distinct pair x and y, there is an $f_{x,y}$ in $\mathcal{F}$ such that*

$$f_{x,y}(x) \neq f_{x,y}(y).$$

Example 7.

If $\mathcal{F}$ has only constant functions, then $\tau_{\mathcal{F}}$ is the trivial topology $\{X, \emptyset\}$ which is not Hausdorff.

Exercise 8.

Prove that $\mathcal{F}$-topology is Hausdorff if and only if $\mathcal{F}$ is total.

Exercise 9.

If $\mathcal{F}$ is separating topologizing family of functions for X, then prove that the map given by (1) is a homeomorphism from X with $\mathcal{F}$-topology onto a topological subspace of $\prod_{f \in \mathcal{F}} Y_f$ with the product topology.

Definition 10.

Let X be a topological space.

(c) The space X is a T_0 space if, for each pair of distinct points in X, at least one has a neighborhood not containing the other.

(d) The space X is a T_1 space if, for each pair of distinct points in X, each has a neighborhood not containing the other.

(e) The space X is a *Hausdorff* or *separated* or T_2 space if, for each pair of distinct points x and y in X, there are disjoint neighborhoods U_x and U_y of x and y respectively.

Definition 11.

Let X be a topological space.

(f) The space X is a *regular* or T_3 space if it is a T_1 space, and for each x in X and each closed subset F of X not containing x there are disjoint open sets U and V such that U is a neighborhood of x and V includes F.

(g) The space X is a *completely regular* or *Tychonoff* or $T_{3\frac{1}{2}}$ space if it is a T_1 space, and for each x in X and each closed subset F of X not containing x there is a continuous[1] function $f: X \to [0, 1]$ such that $f(x) = 0$ and $f(y) = 1$ for each y in F.

(h) The space X is a *normal* or T_4 space if it is a T_1 space, and for each pair of disjoint closed subsets F_1 and F_2 of X there are disjoint open sets U_1 and U_2 that include F_1 and F_2 respectively.

Exercises 12.

Prove the following :

1. *A topological space is a* T_1 *space iff each of its one-element subsets is closed.*
2. $T_4 \Rightarrow T_{3\frac{1}{2}} \Rightarrow T_3 \Rightarrow T_2 \Rightarrow T_1 \Rightarrow T_0$.
   Which implication requires an application of Urysohn's lemma?

Separation Axioms

The spaces $Y_{\mathcal{F}}$ can have any of the separation axioms.

But if $\mathcal{F}$ is not a separating family, then $\mathcal{F}$-topology can have no hope of being even a T_0-space: Suppose that there are distinct elements x and y of X such that

$$f(x) = f(y) \text{ whenever } f \in \mathcal{F}.$$

Every member of the standard subbasis for the $\mathcal{F}$-topology that contains either x and y must contain both, so the same hold for each member of the standard basis for the $\mathcal{F}$-topology.

Hence the $\mathcal{F}$-topology is not even T_0.

Separation Axioms

Even though, the spaces Y_f can have none of the separation axioms, one can have a $\mathcal{F}$-topology satisfying any separation axioms.

Example 13.

Let X be any set and Y be a topological space satisfying none of the separation axioms but having a nonempty proper open subset U.
For each x in X, define $f_x : X \to Y$ by

$$f_x(y) = \begin{cases} \textit{some element } z \textit{ in } U & \textit{when } \; y = x \\ \textit{some element } w \textit{ in } X \backslash U & \textit{when } \; y \neq x. \end{cases}$$

The choice of f_x is plenty.
Let $\mathcal{F} = \left\{ f_x : x \in X \right\}$ and $Y_{f_x} = Y$. Then the $\mathcal{F}$-topology of X satisfies any separation axioms, since each of its one-element sets, and hence each of its subsets is open.

Theorem 14.

Suppose $X, \mathcal{F}$ and Y_f as given above. If each Y_f is T_0, or T_1, or T_2, or T_3, or $T_{3\frac{1}{2}}$, then the $\mathcal{F}$-topology of X satisfies that same separation axiom.

Exercise 15.

The preceding result cannot be extended to normal topological spaces, for topological products of normal spaces need not be normal. Find an example.

However, the extension to metrizable topological spaces does hold, provided that the topologizing family is countable.

Recall :

Theorem 16.

If two topologies on the same set result in the same convergent nets with the same limits for those nets, then the two topologies are the same.

Theorem 17.

If $\mathcal{F}$ is separating, countable and the topology of each Y_f is metrizable, then the $\mathcal{F}$-topology of X is metrizable.

Exercise 18.

The countable hypothesis cannot be omitted. Give an example.

Theorem 19.

If X is a compact topological space and there is a countable separating family of continuous metric-space-valued functions on X, then the topology of X is metrizable.

Dual space of a normed space X with a topology (induced by subspaces of $X^{\#}$)

Let X be a vector space and $X^{\#}$ be the space of all linear functionals (not necessarily continuous) on X.

When X is a normed space, $X^{\#}$ is called the **(algebraic) dual** of X.

The set of all continuous linear functionals on X (with a topology τ) is called the **dual space** of X (with a topology τ) and it is denoted by $(X, \tau)^*$.

Can new function cope up ?

Let $X = [0, 1]$ and $\mathcal{F}$ be a collection of **non-constant** functions defined on $[0, 1]$.

Every constant function is continuous with respect to the $\mathcal{F}$-topology.

Every member of $\mathcal{F}$ is continuous with respect to the $\mathcal{F}$-topology. But some functions (other than from $\mathcal{F}$) may also become continuous with respect to the $\mathcal{F}$-topology.

But when elements in $\mathcal{F}$ are linear, this will not happen, as shown in the following theorem.

Definition 20.

Suppose that X is a vector space with a topology τ such that addition of vectors is a continuous operation from $X \times X$ into X and multiplicaiton of vectors by scalars is a continuous operation from $\mathbb{F} \times X$ into X.

Then τ is a **vector** *or* **linear topology** *for X, and the ordered pair (X, τ) is a* **topological vector space (TVS)** *or a* **linear topological space (LTS)**.

If τ has a basis consisting of convex sets, then τ is a **locally convex topology** *and the TVS (X, τ) is a* **locally convex space (LCS)**

The continuity of the vector space operations in a TVS creates a link between the vector space structure and the topology of the space.

The additional property possessed by an LCS provides each of its points with **a supply of nicely shaped neighbourhoods.**

Dual space of a normed space X with a topology (induced by subspaces of $X^{\#}$)

We shall now discuss duals of topologies of X induced by subspaces of $X^{\#}$.

Theorem 21.

Suppose that X is a vector space and that X' is a subspace of $X^{\#}$. Then the X'-topology of X is a locally convex topology, and the dual space of X with respect to this topology is X'.
That is,

$$(X, \tau_{X'})^* = X'.$$

The following lemma is required to prove the preceding result.

Lemma 22 (a linear algebra result).

Suppose that f and $f_1, f_2, \ldots, f_n$ are linear functionals on the same vector space. Then f is a linear combination of $f_1, f_2, \ldots, f_n$ if and only if

$$ker(f_1) \cap ker(f_2) \cap \cdots \cap ker(f_n) \subseteq ker(f).$$

Connection between local base at points of X and a basis for the topology of X

It is often important to know those open subsets of a topological space (X, τ) which contain a particular point $x \in X$.

The connection between a basis for (X, τ) and the collection of local bases at each $x \in X$ is easily established.

Exercise 23.

If $\mathcal{B}$ is a colllection of subsets of X then $\mathcal{B}$ is a basis for (X, τ) if and only if for each $x \in X$ the family $\mathcal{B}_x = \{B \in \mathcal{B} : x \in B\}$ is a local base at x.

Note that union of all local bases, $\cup_{x \in X} B_x$, forms a basis for topology τ.

Another subbasis and basis of the topology (induced by a subspace of $X^{\#}$)

We have discussed the following :

- The collection
$$\left\{ f^{-1}(U) : f \in \mathcal{F}, U \in \mathcal{F}_f \right\}$$
is the **standard subbasis** for the $\mathcal{F}$-topology.
- The **standard basis** for the $\mathcal{F}$-topology is the collection of all sets that are intersection of finitely many members of this subbasis.

We shall discuss a collection of subsets of X something like "local base".

Another subbasis and basis of the topology (induced by a subspace of $X^{\#}$)

Theorem 24.

Suppose that X is a vector space and that X' is a subspace of $X^{\#}$. For each x in X and each f in X', let

$$B(x,\{f\}) = \{\, y : y \in X,\ |f(y-x)| < 1 \,\}.$$

Similarly, for each x in X and each finite subset A of X', let

$$B(x,A) = \{\, y : y \in X,\ |f(y-x)| < 1 \text{ for each } f \text{ in } A \,\}.$$

Another subbasis and basis of the topology (induced by a subspace of $X^{\#}$)

Theorem 25 (contd.).

Let

$$\mathfrak{S} = \{ B(x, \{f\}) : x \in X, f \in X' \}$$

and let

$$\mathfrak{B} = \{ B(x, A) : x \in X, A \text{ is a finite subset of } X' \}.$$

Then $\mathfrak{S}$ is a subbasis and $\mathfrak{B}$ a basis for the X' topology of X. If U is a subset of X that is open with respect to the X' topology and x_0 is an element of U, then there is a finite subset A_0 of X' such that $B(x_0, A_0) \subseteq U$; that is, the set U includes a basic neighborhood of x_0 that is "centered" at x_0.

Theorem 26.

Suppose that X is a vector space and that X' is a subspace of $X^{\#}$. Let (x_α) be a net in X. Then the following are equivalent.

1. *The net (x_α) is Cauchy with respect to the X'-topology of X.*
2. *For each f in X', the net $(f(x_\alpha))$ is Cauchy in $\mathbb{F}$.*
3. *For each f in X', the net $(f(x_\alpha))$ is convergent in $\mathbb{F}$.*

Theorem 27.

Suppose that X is a vector space and that X' is a subspace of $X^{\#}$. Then a subset A of X is bounded with respect to the X'-topololgy if and only if $f(A)$ is bounded in $\mathcal{F}$ for each f in X'.

References

1. Robert E. Megginson, An Introduction to Banach Space Theory, Springer, 1991.