Nets in Topology

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March 20, 2017

Overview

The **topology induced by a norm** on a vector space is a **very strong topology** in the sense that **it has many open sets.**

Some Advantages (when there are more open sets) :

If we consider a function whose domain is such a space (topology induced by a norm – it has many open sets), the function finds it particularly easy to be continuous.

Example 1.

Let X be a non-empty set. Every function defined on X with discrete topology is continuous.

Overview

Some Disadvantages (when there are more open sets) :

The formulation in terms of open covers always having finite subcovers is really the "right" definition of compactness.

It expresses that the spirit of compactness is that there are not "too many" open sets in the topology, in the sense that a finite number will suffice to cover the space. The trouble is that there are too many open sets, so there are too many open covers and not all of them have finite subcovers.

Example 2.

An infinite discrete space cannot be compact because there are too many open sets (every point is open).

We can think of the term "compact" as literally applying to the topology, not to the underlying point set.

Some Disadvantages (when there are more open sets) :

An infinite dimensional normed space (with the topology induced by the norm) always has so many open sets that its closed unit ball cannot be compact.

Because of this, many familiar facts about finite dimensional normed spaces that are based on the Heine-Borel property cannot be immediately generalized to the infinite-dimensional case.

A topological is said to be Heine-Borel property if every closed and bounded subset of the Euclidean space is compact.

The theorem is essentially equivalent to asserting the "completeness of the real numbers : any nonempty bounded subset of $\mathbb{R}$ has a least upper bound".

Topology Induced by Metric

Let X be a non-empty set.

If τ is a topology induced by the metric d defined on a metric space X, then the open sets are all subsets of X that can be realized as the unions of open balls

$$B(x_0, r) = \{x \text{ in } X : d(x_0, x) < r\}$$

where x_0 in X and $r > 0$.

That is, it is the topology generated by the basis consisting of the set of all open ε-balls in the metric space (X, d).

The topological space which is so induced is also known as the **topological space associated with the (given) metric space.**

The metric space whose metric induces this topology can be said to **give rise to the topological space.**

Overview

One of the main purposes of this course is to study topologies for normed spaces that are in general weaker than the norm topology, in the sense that they have fewer open sets, but that are still strong enough to have useful properties.

In general, topologies are not always induced by metrics, so familiar metric space arguments based on the convergence of sequences cannot be used in their usual form.

However, most of those arguments can be adapted to general topological spaces if sequences are replaced by more general objects called nets, whose behavior is much like that of sequences.

In the lecture, we shall discuss about "nets".

Definition 3.

Let X be a set. A *topology* for X is a collection $\mathfrak{T}$ of subsets of X such that

(1) both X and the empty set belong to $\mathfrak{T}$;

(2) for every subcollection of $\mathfrak{T}$, the union of the elements of the subcollection also belongs to $\mathfrak{T}$;

(3) for every finite subcollection of $\mathfrak{T}$, the intersection of the elements of the subcollection also belongs to $\mathfrak{T}$.

The set X with the topology $\mathfrak{T}$ is called the *topological space* $(X, \mathfrak{T})$, or just the *topological space* X when no confusion can result. The elements of $\mathfrak{T}$ are called *open sets*.

Local basis for τ at a point x of X

Definition 4.

Let $(X, \mathfrak{T})$ be a topological space and let x be an element of X. A *(local) basis for* $\mathfrak{T}$ *at* x is a collection $\mathfrak{B}_x$ of open sets containing x such that every open set containing x includes a member of $\mathfrak{B}_x$.

How small we choose an open set U containing x, we can find a member

$$B \quad \text{of} \quad B_x$$

inside U.

Definition 5.

Let X be a set and let $\mathfrak{B}$ be a collection of subsets of X such that

(1) $\bigcup\{B : B \in \mathfrak{B}\} = X$;

(2) if $B_1, B_2 \in \mathfrak{B}$ and $x \in B_1 \cap B_2$, then there is a B_3 in $\mathfrak{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

Let $\mathfrak{T}$ be the collection of all sets that are unions of subcollections of $\mathfrak{B}$. Then $\mathfrak{T}$ is the *topology generated by the basis* $\mathfrak{B}$.

1. τ is the smallest toplogy containing $\mathcal{B}$.
2. The term "basis" is associated with a topology.

Basis for the topology

Let τ be a topology for a set X and $\mathfrak{B}$ be a collection of subsets of X. The following are equivalent:

1. $\mathfrak{B}$ is a basis for τ.
2. Every member of $\mathfrak{B}$ is open (necessary condition) and every open set is union of members of $\mathfrak{B}$.
3. Every member of $\mathfrak{B}$ is open (necessary condition) and for each x in X, the family
$$\{B : B \in \mathfrak{B}, x \in B\}$$
is a local basis for τ at x.

Definition 6.

Let X be a set and let $\mathfrak{S}$ (Fraktur S; notice its resemblance to σ) be a collection of subsets of X. Let $\mathfrak{B}_{\mathfrak{S}}$ be the collection of all sets that are intersections of finitely many members of $\mathfrak{S}$. Then the *topology generated by the subbasis* $\mathfrak{S}$ is the topology generated by the basis $\mathfrak{B}_{\mathfrak{S}}$.

For any subcollection $\mathfrak{S}$ of the power set $\mathcal{P}(X)$ there is a unqiue topology having $\mathfrak{S}$ as a subbasis.

However, there is no unique basis for a given topology.

Subbasis for the topology

We can start with a fixed topology and find subbasis for that topology, and we can also start with an arbitrary subcollection of the power set $\mathcal{P}(X)$ and form the topology generated by that subcollection.

Example

1. The usual topology on the real numbers $\mathbb{R}$ has a subbasis consisting of all semi-infinite open intervals either of the form

$$(-\infty, a) \text{ or } (a, \infty),$$

where a and b are real numbers :

$$(a, b) = (-\infty, b) \cap (a, \infty) \text{ for } a < b.$$

Subbasis for the topology : Examples

1. A subbasis is formed by taking the subfamily

$$\Big\{(a,b) : a, b \in \mathbb{Q}\Big\}.$$

2. The subbasis consisting of all semi-infinite open intervals of the form

$$(-\infty, a) \quad \text{alone}, \quad a \in \mathcal{R}$$

   does not generate the usual topology.
   The resulting topology does not satisfy the T_1-separation axiom.

3. Once nice fact about subbases is that continuity of a function need only be checked on a subbasis of the range.

   Let $\mathfrak{S}$ be a subbasis for Y. A function $f : X \to Y$ is continuous iff $f^{-1}(U)$ is open in X for each U in $\mathfrak{S}$,

Subbasis for the topology : A result

There is one significant result concerning subbasis, due to James Waddell Alexander II.

Theorem 7 (Alexander Subbasis Theorem).

Let X be a topological space with a subbasis $\mathfrak{S}$. X is compact iff every cover by elements of the subbase $\mathfrak{S}$ has a finite subcover.

Using Alexander Subbasis Theorem, the following results have short proofs:

- Heine-Borel Theorem
- Tychonoff Theorem.

Bases and Subbases

- Bases and subbases "generate" a topology in different ways.
- Every open set is a union of basis elements. The open sets in a topology are all possible unions of basis elements.
- Every open set is a union of finite intersection of subbasis elements. The open sets in a topology are all possible unions of finite intersections of subbasis elements.
- Every basis for a topology is a subbasis for that topology.
- Every member of a basis or subbasis for a topology belongs to that topology.

Exercise 8.

Let $X = \{a, b, c\}$ *and* $\mathfrak{S} = \Big\{\{a, b\}, \{a, c\}\Big\}$.

1. *Find the basis* $\mathfrak{B}$ *generated by* $\mathfrak{S}$.
2. *Find the topology* τ *generated by* $\mathfrak{B}$.
3. *Does the topology* τ *contain the empty set* Φ *?*
4. *Is* $\mathfrak{S}$ *a basis for* τ*?*
5. *Find another subbasis for* τ.

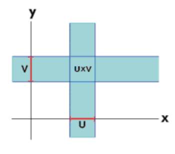
Subbasis but not a basis : Example

Exercise 9.

The set of all infinite rectangles (strips)

$$\left\{\mathbb{R} \times (a, b) : a, b \in \mathbb{R}\right\}$$

forms a subbasis for the usual topology on $\mathbb{R}^2$ but not a basis.



There are open in $\mathbb{R}^2$ which we cannot get from unioning these infinite rectangles together, but they can be made by unioning intersects of these rectangles.

How does the topology generated by a subbasis contain the empty set Φ?

Every element in the empty set is a pink elephant. This is a true statement, since there are no elements in the empty set. These kind of statements are called vacuous statements.

How does the topology generated by a subbasis contain the empty set Φ?

Any statement that says "For all elements in the empty set holds that $\cdots$" is always true, no matter what "$\cdots$" is.

In order for $\Phi \in \tau$, we must apply definition. That is, we must show that for every $x \in \Phi$, "there is a some $B \in \mathfrak{B}$ such that $x \in B \subseteq \Phi$". This is always true.

Indeed, if the statement were false, then there would exist an $x \in \Phi$ that is not in some $B \in \mathfrak{B}$ such that $B \subseteq \Phi$, which is automatically false because there are no elements in the empty set.

Why is subbasis (for a topology) important?

We have the following relation:

$$\mathfrak{S} \subseteq \mathfrak{B} \subseteq \tau.$$

We require more basis elements than subbasis elements.

For this reason, we can take a smaller set as our subbasis, and that sometimes makes proving things about the topology easier.

Definition 10.

Let $\{X_\alpha : \alpha \in I\}$ be a family of topological spaces. Let $\mathfrak{S}$ be the collection of all subsets of the Cartesian product $\prod_{\alpha \in I} X_\alpha$ of the form $\prod_{\alpha \in I} U_\alpha$, where each U_α is open and at most one U_α is not equal to the corresponding X_α. Then the *product topology* of $\prod_{\alpha \in I} X_\alpha$ is the topology generated by the subbasis $\mathfrak{S}$. The *topological product* of the family of topological spaces is the Cartesian product with the product topology.

All definitions are given based on open sets or neighbourhoods of points.

Definition 11.

Let X be a topological space.

(a) A subset of X is *closed* if its complement is open.

(b) A *neighborhood* of a point x in X is an open set that contains x.

Definition 12.

Let X be a topological space.

(c) The space X is a T_0 space if, for each pair of distinct points in X, at least one has a neighborhood not containing the other.

(d) The space X is a T_1 space if, for each pair of distinct points in X, each has a neighborhood not containing the other.

(e) The space X is a *Hausdorff* or *separated* or T_2 space if, for each pair of distinct points x and y in X, there are disjoint neighborhoods U_x and U_y of x and y respectively.

Definition 13.

Let X be a topological space.

(f) The space X is a *regular* or T_3 space if it is a T_1 space, and for each x in X and each closed subset F of X not containing x there are disjoint open sets U and V such that U is a neighborhood of x and V includes F.

(g) The space X is a *completely regular* or *Tychonoff* or $T_{3\frac{1}{2}}$ space if it is a T_1 space, and for each x in X and each closed subset F of X not containing x there is a continuous[1] function $f: X \to [0,1]$ such that $f(x) = 0$ and $f(y) = 1$ for each y in F.

(h) The space X is a *normal* or T_4 space if it is a T_1 space, and for each pair of disjoint closed subsets F_1 and F_2 of X there are disjoint open sets U_1 and U_2 that include F_1 and F_2 respectively.

Exercises 14.

Prove the following :

1. *A topological space is a T_1 space iff each of its one-element subsets is closed.*
2. $T_4 \Rightarrow T_{3\frac{1}{2}} \Rightarrow T_3 \Rightarrow T_2 \Rightarrow T_1 \Rightarrow T_0$.
   Which implication requires an application of Urysohn's lemma?

Definition 15.

Let X be a topological space.

(i) The *relative* or *induced* or *inherited* topology of a subset S of X is the collection of all sets $S \cap U$ such that U is open in X.

(j) The *closure* of a subset S of X, denoted by $\overline{S}$, is the smallest closed set that includes S, that is, the intersection of all closed sets that include S.

(k) The *interior* of a subset S of X, denoted by S°, is the largest open subset of S, that is, the union of all open subsets of S.

(l) The *boundary* of a subset S of X, denoted by ∂S, is the set $\overline{S} \cap \overline{X \setminus S}$, that is, the set $\overline{S} \setminus S^\circ$.

(m) A subset D of X is *dense* in another subset S of X if $D \subseteq S \subseteq \overline{D}$.

(n) A *limit point* or *cluster point* or *accumulation point* of a subset S of X is a point x in X such that each neighborhood of x contains at least one point of S distinct from x, that is, such that $x \in \overline{S \setminus \{x\}}$.

Definition 16.

Let X be a topological space.

(o) A subset S of X is *compact* if, for each collection $\mathfrak{O}$ of open sets whose union includes S, there is a finite subcollection of $\mathfrak{O}$ whose union includes S. That is, the set S is compact if each open covering of S can be thinned to a finite subcovering. The set S is *relatively compact* if its closure is compact.

(p) The space X is *locally compact* if, for each x in X, there is a compact subset K_x of X such that $x \in K_x^\circ$.

Definition 17.

Let X be a topological space.

(q) A subset S of X is *countably compact* if each countable open covering of S can be thinned to a finite subcovering. The set S is *relatively countably compact* if its closure is countably compact.

(r) A subset S of X is *limit point compact* or *Fréchet compact* or has the *Bolzano-Weierstrass property* if each infinite subset of S has a limit point in S. The set S is *relatively limit point compact* if it satisfies the same condition except that the limit point need not be in S.

In the following definitions, sequences are used.

Definition 18.

Let X be a topological space.

(t) A subset S of X is *sequentially compact* if each sequence in S has a convergent subsequence with a limit in S. The set S is *relatively sequentially compact* if it satisfies the same condition except that the limit need not be in S.

(u) The *sequential closure* of a subset S of X is the collection of all elements of X that are limits of sequences whose terms come from S. The set S is *sequentially closed* if it equals its sequential closure.

(v) A subset D of X is *sequentially dense* in another subset S of X if $D \subseteq S \subseteq \overline{D}^s$, where $\overline{D}^s$ is the sequential closure of D.

Exercises 19.

Prove the following :

1. *In a metric space, the properties of compactness, countable compactness, limit point compactness, and sequential compactness are equivalent, as are the corresponding relative properties.*
2. *Compactness implies countable compactness.*
3. *Countable compactness and limit point compactness are equivalent in Hausdorff spaces.*

Sequential testing fails in some topological spaces

Not every topology permits such straightforward sequential testing for continuity and closure.

Some topological spaces have subsets that are sequentially closed but not closed.

Exercise 20.

Let X *be the interval* $[0, 1]$ *with the topology given by declaring that a subset of* X *is open if it does not contain* 0 *or its complement is countable.*

Verify that X *is a Hausdorff topological space.*

Show that the subset $(0, 1]$ *of* X *is sequentially closed but not closed.*

Sequential testing fails in some topological spaces

Consider a topological space having subsets that are sequentially closed but not closed.

Though it might seem that sequential methods useful in metric spaces must be abandoned when working with topologies of this sort, many of those methods extend with very little modification to all topological spaces **if sequences are replaced by nets.**

Overview for nets

In a metric space,

- **Openness :** A is open iff no sequence with terms outside of A has a limit inside A.
- **Continuity :** For any topological spaces X and Y, a function $f : X \to Y$ is continuous iff f preserves convergence (for each x in X, whenever $x_n \to x$ in X, also $f(x_n) \to f(x)$ in Y).
- **Compactness :** A metric space is compact iff every sequence has a converging subsequence (sequentially compact).

What about these equivalences in a general topological space?

In general, these equivalences may fail.

Sequences do not fully encode all information about

- open sets.
- functions between topological spaces.
- compact sets.

Misconceptions (in our mind) :

In a general topological space,

- compactness would always be equal to sequence compactness.
- If f preserves convergence, then f is continuous.

Overview

- It will be shown that how sequences might fail to characterize topological properties such as openness, continuity and compactness.
- Nets will be defined and it will be shown that how nets succeed where sequences fail.
- Nets allow analogue results to hold in the context of topological spaces that do not necessarily have a first countable basis.

Overview

Convergence of sequences doesn't give us full information on the topology.

For example, the discrete topology and the countable complement topology on an uncountable set X have the same converging sequences (eventually constant sequences), but the discrete topology is stictly finer (bigger) then the countable complement topology.

The discrete topology is sequential, but the countable complement topology contains sequentially open sets which are not open.

Overview

Convergence of sequences works fine when the space is first countable because a countable basis at a point allows us to **approach** that point nicely with a sequence. However, if a point x does not have a countable basis, then sequences might not succeed in getting **close to** the point x, eventually in every neighbourhood of x.

Sequences fall short in two respects : They are **too short** and **too thin**.

Sequentially Open

In a topological space X, a set A is **open** if every $a \in A$ has a neighbourhood contained in A.

A is **sequentially open** if no sequence with terms outside of A (in $X \backslash A$) has a limit in A.

Theorem 21.

In a topological space X, if A is open, then A is sequentially open.

Outline : Suppose A is open. Take any (x_n) in $X \backslash A$ and any $y \in A$.
Claim : y cannot be a limit point of (x_n) because $X \backslash A$ is closed.

Theorem 22.

If X is a metric space, then the two notions of open and sequentially open are equivalent.

Outline : Suppose A is open. Take any (x_n) in $X \backslash A$ and any $y \in A$.
Claim : y cannot be a limit point of (x_n) because $X \backslash A$ is closed.

Converse, suppose A is not open. Then there exist $y \in A$ such that every neighbourhood of y intersects $X \backslash A$. The sequence $x_n \in (X \backslash A) \cap B(y, 1/n)$. Then (x_n) in $X \backslash A$ converges to y in A, a contradiction.

Theorem 23.

Let X be a topological space. Then $A \subseteq X$ is sequentially open iff every sequence with a limit in A has all but finitely many terms in A.

Outline : Using defintion of sequentially open, we shall prove the following :

$A \subseteq X$ is not sequentially open iff there exists a sequence with a limit in A has finitely many terms in A.

Definition 24.

A topological space is **sequential** *when "any set A is open iff A is sequentially open".*

Not every topological space is sequential – Example

Let X be an infinite set. We define the **countable-complement topology** on X by declaring the empty set to be open, and a non-empty subset U of X to be open if $X \setminus U$ is countable.

$$\tau_{cc} = \left\{ U \subseteq X : X \setminus U \text{ is countable } \right\} \cup \{\emptyset\}.$$

Exercise 25.

1. *Prove that τ_{cc} is a topology.*
2. *If X is countable, then the countable-complement topology is just the discrete topology, as the complement of any set is countable and thus open.*
3. *A subset A of X is closed in the topology τ_{cc} iff $A = X$ or A is countable.*

The topology τ_{cc} has the following additional properties :

- The countable-complement topology on an uncountable set gives an example of a topological space that is not weakly countably compact (but it is pseudo compact).
- Suppose X has at least 2 points. Then (X, τ_{cc}) is connected iff X is an uncountable set.
- The real numbers with the topology τ_{cc} fail to have the Bolzano-Weierstrass property, and hence $(\mathbb{R}, \tau_{cc})$ is not compact.

Not every topological space is sequential – Example

Let X be an uncountable set. The **countable-complement topology** on X

$$\tau_{cc} = \Big\{ U \subseteq X : X \backslash U \text{ is countable } \Big\} \cup \{\emptyset\}.$$

is not sequential.

Convergence sequences in the topology τ_{cc} :

Suppose $(x_n) \subseteq X \backslash A$ has a limit y. Then for every neighbourhood U of y, we can find N such that $x_n \in U$ for all $n \geq N$. We now consider the (special) neighbourhood of y

$$(\mathbb{R} \backslash \{x_n\}) \cup \{y\}.$$

Note that $(\mathbb{R} \backslash \{x_n\})$ may not be a neighbourhood of y (what happens if $x_n = y$ for some n?). This neighbourhood should contain x_n for n large enough. Hence the sequence is eventually constant.

Not every topological space is sequential – Example

Claim : X has a subset which is sequentially open but not open.

Consider $\{x\}$. **Claim :** $\{x\}$ is sequentially open. Suppose $\{x\}$ is not sequentially open. Then there exists a sequence (x_n) in $X \backslash A$ such that x_n converges to some $y \in A$, a contradiction to choice of x_n in $X \backslash A$.

There is nothing special about $\{x\}$. This above argument is true for any subset A of X. Thus every subet of X is sequentially open.

Since X is uncountable, $\tau_{cc} \subsetneq \tau_{discrete} = \mathcal{P}(X)$. Then X has a subset which is not open.

Every first countable space is sequential.

Still, a large class of topological spaces is sequential.

Definition 26.

A **countable basis at a point** x *is a countable set*

$$\left\{ U_n : n \in \mathbb{N} \right\}$$

of neighbourhoods of x *such that for any neighbourhood* V *of* x*, there exists an* $n \in \mathbb{N}$ *such that* $U_n \subseteq V$.

A topological space is **first countable** *if every point has a countable basis.*

Every first countable space is sequential.

Theorem 27.

Every first countable space X (and hence every metric space) is sequential.

Outline : Let A be a sequentially open. Suppose A is not open. Then there exists $y \in A$ such that every neighbourhood of y intersects $X \setminus A$.

Let $\{U_n\}$ be a countable basis at y. For every n, we have $x_n \in X \setminus A \cap (\cap_{i=1}^n U_i)$.

Claim : (x_n) converges to $y \in A$, a contradiction.

Continuous functions from sequential spaces into another topological spaces

Sequential spaces are also exactly those spaces X where sequences can correctly define continuity of functions from X into another topological space.

Theorem 28.

The following are equivalent for any topological space X :

1. *X is sequential ;*
2. *for any topological space Y and function $f : X \to Y$, f is continuous iff f preserves convergence.*

Definition 29.

A *directed set* is a nonempty set I with a relation $\preceq$ such that

(1) $\alpha \preceq \alpha$ whenever $\alpha \in I$;

(2) if $\alpha \preceq \beta$ and $\beta \preceq \gamma$, then $\alpha \preceq \gamma$;

(3) for each pair α, β of elements of I there is a $\gamma_{\alpha,\beta}$ in I such that $\alpha \preceq \gamma_{\alpha,\beta}$ and $\beta \preceq \gamma_{\alpha,\beta}$.

That is, a directed set is a nonempty preordered set that satisfies (3). A *net* or *Moore-Smith sequence* in a set X is a function from a directed set I into X. The set I is the *index set* for the net.

Some sources require that the preorder in the preceding definition be a partial order; that is, that the following additional requirement be included in the definition:

(4) $\alpha = \beta$ whenever $\alpha \preceq \beta$ and $\beta \preceq \alpha$.

The theory of nets can be developed either with or without this additional axiom.

Observations

On conditions (2) and (3) in definition of net

- We don't require that a pair of elements has a **least upper bound**, we just require that some upper bound exists.
- Every finite subset of I has a upper bound in I.

History

- E.H. Moore and H.L. Smith introduced nets in 1922 as the basis for a general theory of limits.
- Mauro Picone devised the same theory independently in a book that appeared the next year.
- The term net was actually first used by J.L. Kelley in a 1950 paper on topological convergence. The terminology was not Kelley's invention, though. Kelley has wanted to call such an object a "**way**".
- After some prodding by Kelley, Norman Steenrod suggested the term "net" as a substitute for "way".

Examples for Nets

For each net in a topological space X, there corresponds a directed set I which gives a function $f : I \to X$.

The α^{th} term $f(\alpha)$ of the net is often denoted by x_α, and the entire net is often denoted by $(x_\alpha)_{\alpha \in I}$.

1. Every sequence is a net, with the directed set being $\mathbb{N}$ in its natural order.
2. The set $\mathbb{R}$ with its natural order is a directed set, so this order makes every function with domain $\mathbb{R}$ into a net.
   By analogy with sequences, it is said that x_α precedes x_β in a net when $\alpha \preceq \beta$.
   - Do these nets have first terms?
   - How many predecessors are there for each term in a net?

Example for a net

The set $\mathbb{R}^2$ can be made into a directed set by declaring that

$$(\alpha_1, \beta_1) \preceq (\alpha_2, \beta_2)$$

whenever $\alpha_1 \leq \alpha_1$.

If $x_{(\alpha,\beta)} = \alpha + \beta$ for each (α, β) in $\mathbb{R}^2$, then $(x_{(\alpha,\beta)})$ is a net in $\mathbb{R}^2$.

Does $\preceq$ satisfy the "transitive property"?

Example for a net

We have seen an example that nets may not have first terms.

Exercise 30.

Give an example of a net which illustrates the following :

- *The index set for a net can be finite.*
- *Nets can have last terms.*
- *Nets can have multiple "first" terms, that is, terms not preceded by other terms.*
- *The index set for a net need not be a chain.*

It can be concluded that several important ways in which nets can differ from sequences.

Here is a type of net that is useful in many topological arguments.

Suppose that X is a topological space and that $x \in X$.

Let I be the collection of all neighborhoods of x with the relation $\preceq$ given by declaring that

$$U \preceq V$$

when $U \supseteq V$ (the direction is given by **reverse inclusion**).

Then I is a directed set. If $x_U \in U$ for each U in I, then (x_U) is a net in X.

A sequence in a topological space converges to an element of the space if, for every neighbourhood of that element, all terms of the sequence from some term onward lie in that neighborhood.

This definition generalizes immediately to nets.

Definition 31.

Let $(x_\alpha)_{\alpha \in I}$ be a net in a topological space X and let x be an element of X. Then (x_α) **converges** *to x, and x is called a* **limit** *of (x_α), if, for each neighbourhood U of x, there is an α_U in I such that $x_\alpha \in U$ whenever $\alpha_U \preceq \alpha$.*

Convergence of nets – Examples

The set $\mathbb{R}$ with its natural order is a directed set, so this order makes every function with domain $\mathbb{R}$ into a net.

Consider a net of a bounded increasing function.

Where does the net converge?

Convergence of nets – Examples

Let I be a three-element set $\left\{u, v, w\right\}$. Define $\preceq$ on I by letting these be all of the corresponding relations :

$$\alpha \preceq \alpha \quad \text{for each} \quad \alpha \in I$$

and $u \preceq w$; and $v \preceq w$.

Define a net (x_α) in $\mathbb{R}$ with index set I by letting $x_u = 0, x_v = \pi$, and $x_w = -3$.

Where does the net (x_α) converge?

Convergence of nets – Examples

Let I be the collection of all neighborhoods of x with the relation $\preceq$ given by declaring that

$$U \preceq V$$

when $U \supseteq V$ (the direction is given by **reverse inclusion**).

Then I is a directed set. If $x_U \in U$ for each U in I, then (x_U) is a net in X.

Where does the net (x_U) converge?

Only subbasic neighbourhoods really need to be checked

Theorem 32.

Suppose that $\mathfrak{S}$ is a subbasis for the topology of a topological space X, that $(x_\alpha)_{\alpha \in I}$ is a net in X, and that $x \in X$. Then $x_\alpha \to x$ if and only if the following is true : For every member U of $\mathfrak{S}$ that contains x, there is an α_U in I such that $x_\alpha \in U$ whenever $\alpha_U \preceq \alpha$.

A useful consequence :

If (x_α) is a net and $\Big\{P_1, P_2, \ldots, P_n\}$ is a finite collection of properties for the terms of the net such that P_j holds from some corresponding net index value α_j onward, then there is an index value α such that $P_1, P_2, \ldots, P_n$ all hold from α onward.

For nets in a topological product, convergence is equivalent to coordinatewise convergence

Theorem 33.

Let $\left\{X^{(\alpha)} : \alpha \in I\right\}$ be a family of topological spaces and let X be their topological product. Suppose that $(x_\beta)_{\beta \in J}$ is a net in X and x is a member of X. Then $x_\beta \to x$ if and only if $x_\beta^{(\alpha)} \to x^{(\alpha)}$ for each α in I.

Nets characterize Hausdorff spaces

- In Hausdorff spaces, every converging sequence has a unique limit.
- Sequences can also have unique limits in spaces that are not Hausdorff.

Example 34.

Consider an uncountable set X with the countable complement topology.
We have seen that limits are unique.
But the space is not Hausdorff.

Nets characterize Hausdorff spaces

In a Hausdorff space, convergent sequences have unique limits. On the contrary, nets do succeed in exactly characterizing Hausdorff spaces.

The corresponding statement for nets actually characterizes Hausdorff spaces among all topological spaces.

Theorem 35.

A topological space X is a Hausdorff space if and only if each convergent net in X has only one limit.

Point is in the closure of a set

In a metric space, a point is in the closure of a set if and only if some sequence from the set converges to that point.

If sequences are replaced by nets, then this remains true for arbitrary topological spaces.

Theorem 36.

Let S be a subset of a topolgical space X and let x be an element of X. Then $x \in \overline{S}$ iff some net in S converges to x.

Point is in the closure of a set

A point x is a limit point of S iff $x \in \overline{S \setminus \{x\}}$ iff there is a net in $S \setminus \{x\}$ converging to x.

We know that a set in a topological space is closed iff it includes its closure. The following result generalizes the fact that sets in metric spaces are closed exactly when they are sequentially closed.

Theorem 37.

A subset S of a topolgical space is closed iff S contains every limit of every net whose terms lie in S.

Equivalence of continuity and sequential continuity

The following result is a generalization of the equivalence of continuity and sequential continuity for functions from a metric space into a topological space.

Theorem 38.

Let X and Y be topological spaces and let f be a functin from X into Y.

1. *The function f is continuous at the point x_0 of X iff $f(x_\alpha) \to f(x_0)$ whenever (x_α) is a net in X converging to x_0.*
2. *The function f is continuous on X iff $f(x_\alpha) \to f(x_0)$ whenever (x_α) is a net in X converging to an x in X.*

Theorem 39.

If two topologies on the same set result in the same convergent nets with the same limits for those nets, then the two topologies are the same.

Under the hypotheses, the identity map on the space, treated as a map between the two topological spaces in question, is continuous in each direction and so is a homeomorphism.

Conclusion

- Nets are one of the many tools used in topology to generalize certain concepts that may only be general enough in the context of metric spaces.
- Nets are defined to overcome the shortcomings of sequences.
- Nets gneralize sequences, but they can go both **deeper** and **wider** then sequences.
- Sequences associate a point to every natural number. Nets are more general, as they can associate a point to every element to a directed set.

References

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