Zabreiko's lemma and four fundamental theorems of functional analysis

P. Sam Johnson

NITK, Surathkal, India

Outline of the talk

There are no doubts that open mapping theorem, closed graph theorem, bounded inverse theorem, uniform boundedness principle are the fundamental theorems of functional analysis. All of them are similar in the sense that they explicitly or implicitly use the Baire category theorem, but in the details they are different. However, all these theorems can be formulated as continuity of certain seminorms defined on a normed space.

We can achieve this by proving a lemma, called, Zabreiko's lemma to prove the aforementioned theorems in the lecture.

Notations

Before we start, let us see the notations first.

$\mathbb{K}$	the field of real or complex scalars
ℓ_1	the set of absolutely summable sequences
ℓ_2	the set of square summable sequences
ℓ_∞	the set of bounded sequences
c_0	the set of convergent sequences coverging to 0
B_X	the closed unit ball in X
S_X	the unit sphere in X
$\overline{M}$	the closure of M
X^*	the dual of X
T^*	the adjoint of T
$\|T\|$	a norm of the operator T
$\mathcal{B}(X, Y)$	the space of bounded linear operators from X into Y

Recall : Open Mapping Theorem

Proposition 1.

Let T be a bounded linear operator between normed spaces X and Y. If T is an open map, then T is onto.

The converse of Proposition 1 holds good provided X and Y are Banach and T is bounded. The following result is the open mapping theorem, also known as the **interior mapping principle.**

Theorem 2 (J. Schauder, 1930).

Let X and Y be Banach and $T \in \mathcal{B}(X, Y)$. If T is onto, then T is open.

Theorem 3 (an application).

Let X and Y be normed spaces and $T : X \to Y$ be linear. Then T is an open map iff there exists some $\gamma > 0$ such that for every $y \in Y$, there is some $x \in X$ with $Tx = y$ and $\|x\| \leq \gamma \|y\|$.

Theorem 4 (Bounded Inverse Theorem, S. Banach, 1929).

Let X and Y be Banach spaces and $T \in \mathcal{B}(X, Y)$ be bijective. Then $T^{-1} \in \mathcal{B}(Y, X)$.

The preceding theorem is sometimes called the **inverse mapping theorem**.

The following result is an application of the bounded inverse theorem which is of a positive nature.

Theorem 5 (Two-Norm Theorem).

Let X be a Banach space with respect to $\|.\|_1$ and $\|.\|_2$. Then the topologies generated by these two norms are the same if either is stronger than other.

Recall : Uniform Boundedness Principle

Theorem 6 (H. Hahn, 1922; S. Banach 1922; T.H. Hildebrandt, 1923; S. Banach and H.Steinhaus, 1927).

Let X be a Banach space, Y be a normed space and $\mathcal{A} \subseteq \mathcal{B}(X, Y)$. If $\mathcal{A}$ is pointwise bounded, then $\mathcal{A}$ is uniformly bounded. That is, when X is Banach, uniform boundedness and pointwise boundedness are same.

Geometrically, the uniform boundedness theorem says that either each $A \in \mathcal{A}$ maps a bounded subset of a Banach space X into a fixed ball in Y or there is some $x \in X$ such that no ball in Y contains all Ax with $A \in \mathcal{A}$. The choice of such x is dense in X. One of the many interesting applications of the uniform boundedness principle is to show the existence of continuous functions on $[-\pi, \pi]$ whose Fourier series does not converge at each point of a dense set in $[-\pi, \pi]$: There exists a dense subset D of $X = \{x \in C[-\pi, \pi] : x(\pi) = x(-\pi)\}$ such that the Fourier series of every $x \in D$ diverges at every rational number in $[-\pi, \pi]$.

Recall : Closed Graph Theorem

Definition 7.

Let X and Y be normed spaces and X_0 be a subspace of X. A linear operator $T : X_0 \subseteq X \to Y$ is a ***closed operator*** *if for every sequence (x_n) in X_0 such that $x_n \to x$ and $Tx_n \to y$ for some $x \in X$ and $y \in Y$, then $x \in X_0$ and $Tx = y$.*

Theorem 8 (A characterization of closed linear operator).

Let X and Y be normed spaces and X_0 be a subspace of X. A linear operator $T : X_0 \subseteq X \to Y$ is a closed operator iff its graph $G(T) = \{(x, Tx) : x \in X_0\}$ is a closed subspace of $X \times Y$ with respect to the norm on the product space, $\|(x, y)\| = \|x\|_X + \|y\|_Y$ or any equivalent norm.

Theorem 9 (Closed Graph Theorem, S. Banach, 1932).

Let X and Y be Banach spaces and $T : X \to Y$ be linear. Then T is bounded iff T is closed.

A weaker version of the Baire category theorem

We shall now prove the Zabreiko's lemma. To prove it, we need a result which is a weaker version of the Baire category theorem. Much of the theory of Banach spaces is based on three related results, called the open mapping theorem, uniform boundedness principle, and closed graph theorem, whose conclusions do not hold for arbitrary normed spaces.

The general plan of attack is to derive each of them from a result called Zabreiko's Lemma that is itself a straightforward consequence of the following theorem, which is a weak version of the Baire category theorem.

Theorem 10.

Every closed, convex, absorbing subset of a Banach space includes a neighborhood of the origin.

Note that there are incomplete normed spaces having closed, convex, absorbing subsets with empty interiors.

Absorbing Set

Definition 11.

Let A be a subset of a vector space X. The set A is ***absorbing*** *in X if for each $x \in X$, there is positive number s_x such that $x \in tA$ whenever $t > s_x$.*

Exercise 12.

Show that absorbing sets always contain 0.

We require two notions, namely, seminorm and countably additive, to see the statement of Zabreiko's lemma.

Seminorm

Definition 13.

*Let X be a linear space. A **seminorm or prenorm** on X is a real-valued function p on X such that the following conditions are satisfied by all members x and y of X and each scalar α :*

1. $p(\alpha x) = |\alpha|\ p(x)$;
2. $p(x+y) \leq p(x) + p(y)$.

- If X and Y are normed spaces and T is a linear operator from X into Y, then the function $x \mapsto \|Tx\|$ from X into $\mathbb{R}$ is a seminorm on X, called the **seminorm induced by** T.
- Norm is always a seminorm, and is in fact just the seminorm induced by the identity operator on the space.
- $p(0) = 0$, since $p(0) = p(0.0) = 0.p(0)$.
- p is a nonnegative-real-valued function, since $0 = p(0) \leq p(x) + p(-x) = 2p(x)$ for each X.

Countably Subadditive

Definition 14.

A function f from a normed space X into the non-negative reals is ***countably subadditive*** *if*

$$f\left(\sum_{n=1}^{\infty} x_n\right) \leq \sum_{n=1}^{\infty} f(x_n)$$

for each convergent series $\sum_{n=1}^{\infty} x_n$ in X.

For example, the norm of a normed space X is always countably subadditive by the following exercise.

Exercise 15.

If $\sum_n x_n$ is a finite or infinite sum in a normed space X, then $\|\sum_n x_n\| \leq \sum_n \|x_n\|$.

Proposition 16.

Let p be a seminorm on a normed space X. If p is continuous, then p is countably subadditive.

Proof of the proposition can be seen by letting $\sum_n x_n$ be a convergent series in X and letting m tend to infinity in the inequality

$$p\Big(\sum_{n=1}^{m} x_n\Big) \leq \sum_{n=1}^{m} p(x_n).$$

Converse of Proposition 16

If p is countably subadditive, then p must be continuous provided that X is a Banach space. That is the content of Zabreiko's lemma.[1]

Lemma 17 (Zabreiko's lemma).

Every countably subadditive seminorm on a Banach space is continuous.

[1] Petr P. Zabreiko, *A theorem for semiadditive functionals, Funktsional. Anal. i Prilozhen. 3 (1969), 86-88 (in Russian).*

Proof of Zabreiko's lemma

Let p be a countably subadditive seminorm on a Banach space X. If p is continuous at 0 and x is an element of X, then one can show that

$$|p(x) - p(y)| \leq p(x - y) = |p(x - y) - p(0)|$$

whenever $y \in X$, which implies the continuity of p at x.

Thus, Zabreiko's lemma will be proved once it is shown that p is continuous at 0.

Let $G = \{x : x \in X, p(x) < 1\}$, the "open unit ball" for p. If $t > 0$, then $tG = \{x : x \in X, p(x) < t\}$ by an application of property $p(\alpha x) = |\alpha|\ p(x)$ in Definition of seminorm, and so $x \in tG$ whenever $x \in X$ and $t > p(x)$.

Thus the set G is absorbing.

Proof (contd...)

If $x, y \in G$ and $0 < t < 1$, then

$$p(tx + (1-t)y) \leq tp(x) + (1-t)p(y) < 1,$$

so $tx + (1-t)y \in G$, which shows that G is convex.

Therefore $\overline{G}$ is closed, convex, and absorbing, and so by Theorem 10, $\overline{G}$ includes an open ball U centered at 0 with some positive radius ε.

If there is a positive real number s such that $p(x) < s$ whenever $\|x\| < \varepsilon$, then $p(x) < t$ whenever $t > 0$ and $\|x\| < s^{-1}t\varepsilon$, which would imply the continuity of p at 0. Thus, the proof will be complete once such an s is found.

Proof (contd...)

Fix an x in X such that $\|x\| < \varepsilon$. Since $x \in U \subseteq \bar{G}$, there is an x_1 in G such that $\|x - x_1\| < 2^{-1}\varepsilon$. Since

$$x - x_1 \in 2^{-1}U \subseteq 2^{-1}\overline{G} = \overline{2^{-1}G},$$

there is an x_2 in $2^{-1}G$ such that $\|x - x_1 - x_2\| < 2^{-2}\varepsilon$. Similarly, there is an x_3 in $2^{-2}G$ such that $\|x - x_1 - x_2 - x_3\| < 2^{-3}\varepsilon$. Continuing in this way yields a sequence (x_n) such that $x_n \in 2^{-n+1}G$ and $\|x - \sum_{j=1}^{n} x_j\| < 2^{-n}\varepsilon$ for each positive integer n. It follows that $p(x_n) < 2^{-n+1}$ for each n and that $x = \sum_n x_n$, and so the countable subadditivity of p implies that

$$p(x) = p\left(\sum_n x_n\right) \leq \sum_n p(x_n) < 2.$$

Letting $s = 2$ completes the proof.

Proof of four fundamental theorems of functional analysis by using Zabreiko's lemma

Now the proof of the fundamental theorems becomes an easy exercise. We shall see them one by one.

Detailed solutions can be found in Theorems 1.6.5, 1.6.6, 1.6.9 and 1.6.11 in a book by Robert E. Megginson. [2]

[2] Robert E. Megginson, *An Introduction to Banach Space Theory*, Graduate Texts in Mathematics.

Open Mapping Theorem

Theorem 18 (J. Schauder, 1930).

Let X and Y be Banach and $T \in \mathcal{B}(X, Y)$. If T is onto, then T is open.

Proof. Let T be a bounded linear operator from a Banach space X onto a Banach space Y. Suppose that the image under T of the open unit ball U of X is open. Let V be an open subset of X. If $x \in V$, then $x + rU \subseteq V$ for some positive r, and so $T(V)$ includes the neighborhood $Tx + rT(U)$ of Tx. It follows that $T(V)$ is open. Thus, the theorem will be proved once it is shown that $T(U)$ is open.

For each y in Y, let $p(y) = \inf\{\|x\| : x \in X, Tx = y\}$. If $y \in Y$ and α is a nonzero scalar, then $\{x : x \in X, Tx = \alpha y\} = \{\alpha x : x \in X, Tx = y\}$, and so

$$\begin{aligned} p(\alpha y) &= \inf\{\|\alpha x\| : x \in X, Tx = y\} \\ &= |\alpha| \cdot \inf\{\|x\| : x \in X, Tx = y\} = |\alpha| p(y). \end{aligned}$$

Proof (contd...)

Since $p(0y) = 0 = |0|p(y)$ whenever $y \in Y$, it follows that $p(\alpha y) = |\alpha|p(y)$ for each scalar α and each y in Y. Now let $\sum_n y_n$ be a convergent series in Y.

The goal is to show that $p\left(\sum_n y_n\right) \leq \sum_n p(y_n)$, so $\sum_n p(y_n)$ can be assumed to be finite. Fix a positive ε. Let (x_n) be a sequence in X such that $Tx_n = y_n$ and $\|x_n\| < p(y_n) + 2^{-n}\varepsilon$ for each n. Then $\sum_n \|x_n\| < \sum_n p(y_n) + \varepsilon$, a finite number. Since X is a Banach space, the absolutely convergent series $\sum_n x_n$ converges.

Now $T(\sum_n x_n) = \sum_n Tx_n = \sum_n y_n$, and so

$$P\left(\sum_n y_n\right) \leq \left\|\sum_n x_n\right\| \leq \sum_n \|x_n\| < \sum_n p(y_n) + \varepsilon.$$

Proof (contd...)

Therefore $p(\sum_n y_n) \leq \sum_n p(y_n)$ since ε is an arbitrary positive number, and so p is countably subadditive.

This also implies that $p(y_1 + y_2) \leq p(y_1) + p(y_2)$ whenever $y_1, y_2 \in Y$, as can be seen by letting $y_n = 0$ when $n \geq 3$.

Thus, the function p is a countably subadditive seminorm on Y, and so is continuous by Zabreiko's lemma. Finally,

$$T(U) = \{y : y \in Y, Tx = y \text{ for some } x \text{ in } U\} = \{y : y \in Y, p(y) < 1\},$$

so $T(U)$ is open.

Bounded Inverse Theorem

Theorem 19 (Bounded Inverse Theorem, S. Banach, 1929).

Let X and Y be Banach spaces and $T \in \mathcal{B}(X, Y)$ be bijective. Then $T^{-1} \in \mathcal{B}(X, Y)$.

Proof. For each $y \in Y$, let $p(y) = \|T^{-1}x\|$.

Proof follows from the proof of the Theorem 18.

Uniform Boundedness Principle

The following result is often called the Banach-Steinhaus theorem, since a proof of it appeared in a 1927 paper by Stefan Banach and Hugo Steinhaus. The result in its full generality for Banach spaces was actually first published in 1923 by T.H.Hildebrandt, though special forms of it had previously appeared in a 1922 paper by Hans Hahn as well as in Banach' doctoral thesis, also published in 1922. In particular, Hahn's proof of the result for the special case in which the family of mappings is a sequence of bounded linear functionals can easily be modified to prove the more general theorem stated here.

Theorem 20 (H. Hahn, 1922; S. Banach 1922; T.H. Hildebrandt, 1923; S. Banach and H.Steinhaus, 1927).

Let X be a Banach space, Y be a normed space and $\mathcal{A} \subseteq \mathcal{B}(X, Y)$. If $\mathcal{A}$ is pointwise bounded, then $\mathcal{A}$ is uniformly bounded. That is, when X is Banach, uniform boundedness and pointwise boundedness are same.

Proof of the theorem

Let $p(x) = \sup\{\|Tx\| : T \in \mathcal{A}\}$ for each x in X, and suppose that p is finite-valued. Notice that $p(\alpha x) = |\alpha| p(x)$ for each x in X and each scalar α. If $\sum_n x_n$ is a convergent series in X and $T \in \mathcal{A}$, then

$$\left\| T\left(\sum_n x_n\right)\right\| = \left\|\sum_n Tx_n\right\| \leq \sum_n \|Tx_n\| \leq \sum_n p(x_n),$$

from which it follows that $p(\sum_n x_n) \leq \sum_n p(x_n)$. In particular, whenever $x_1, x_2 \in X$, letting $x_n = 0$ when $n \geq 3$ shows that $p(x_1 + x_2) \leq p(x_1) + p(x_2)$, so p is a countably subaddtive seminorm on X.

Therefore p is continuous, so there is some positive δ such that $p(x) \leq 1$ whenever $\|x\| \leq \delta$. It follows that $p(x) \leq \delta^{-1}$ whenever $x \in B_X$, and therefore that $\|Tx\| \leq \delta^{-1}$ whenever $T \in \mathcal{A}$ and $x \in B_X$, that is, that $\|T\| \leq \delta^{-1}$ for each T in $\mathcal{A}$.

Closed Graph Theorem

Theorem 21 (Closed Graph Theorem, S. Banach, 1932).

Let X and Y be Banach spaces and $T : X \to Y$ be linear. Then T is bounded iff T is closed.

Proof. Let $p(x) = \|Tx\|$ for each x in X. It is enough to prove that p is continuous, for then there would be a neighborhood U of 0 such that the set $p(U)$ is bounded, which would in turn imply that $T(U)$ is bounded, and that would imply the continuity of T. Since p is a seminorm on X, an application of Zabreiko's lemma will finish the proof once it is shown that p is countably subadditive. Let $\sum_n x_n$ be a convergent series in X. The proof will be finished once it is shown that $\|T(\sum_n x_n)\| \leq \sum_n \|Tx_n\|$, so it may be assumed that $\sum_n \|Tx_n\|$ is finite, which together with the completeness of Y implies that the absolutely convergent series $\sum_n Tx_n$ converges.

Proof (contd...)

Since $\sum_{n=1}^{m} x_n \to \sum_n x_n$ and

$$T\left(\sum_{n=1}^{m} x_n\right) = \sum_{n=1}^{m} Tx_n \to \sum_n Tx_n$$

as $m \to \infty$, it follows from the hypotheses of the theorem that $\sum_n Tx_n = T(\sum_n x_n)$.

Therefore

$$\left\|T\left(\sum_n x_n\right)\right\| = \left\|\sum_n Tx_n\right\| \leq \sum_n \|Tx_n\|,$$

which shows that p is countably subadditive and finishes the proof.

References

Robert E. Megginson, *An Introduction to Banach Space Theory*, Graduate Texts in Mathematics.

Petr P. Zabreiko, *A theorem for semiadditive functionals*, Funktsional. Anal. i Prilozhen. 3 (1969), 86-88 (in Russian).

Petr P. Zabreiko, *A theorem for semiadditive functionals*, Functional analysis and its applications 3 (1), 1969, 70-72 (English translation).