Uniform Continuity

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Overview

The following are well-known concepts in analysis.

- uniform continuity ;
- Lipschitz ;
- contraction.

These are three more concepts of **global continuity**, all of them stronger than continuity and each one stronger than the one before it.

In the lecture, we discuss uniform continuity, Lipschitz and contraction.

Michael-147

Part - 1

Continuity

Let us recall the definition of continuity between metric spaces.

Let (X, d) and (Y, ρ) be metric spaces.

Definition 1.

A function $f : X \to Y$ is said to be **continuous at a point** $c \in X$ *if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $x \in X$,*

$$d(x, c) < \delta \quad \text{implies} \quad \rho(f(x), f(c)) < \varepsilon.$$

Note that the choice of δ depends on ε and on the point c in X. If this point $c \in X$ is changed and the same $\varepsilon > 0$ is given, generally $\delta > 0$ can be different from the original one.

Let A be a non-empty subset of X. If f is continuous at every point in the set A, then we say that f is **continuous on** A.

Examples

The set of real numbers $\mathbb{R}$ equipped with the metric of absolute distance $d(x, y) = |x - y|$ defines the **standard metric space of real numbers** $\mathbb{R}$**.** We now see some examples of continuous functions $f : A \subseteq \mathbb{R} \to \mathbb{R}$.

Example 2.

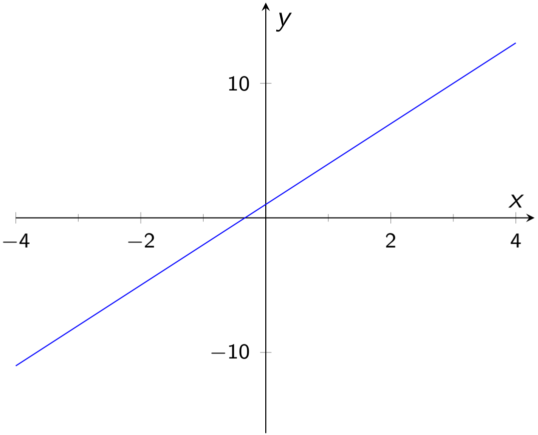
To show that $f(x) = 3x + 1$ *is continuous at an arbitrary point* $c \in \mathbb{R}$*, we must argue that* $|f(x) - f(c)|$ *can be made arbitrarily small for values of* x *near* c*. Now,* $|f(x) - f(c)| = |(3x + 1) - (3c + 1)| = 3|x - c|$*, so, given* $\varepsilon > 0$*, we choose* $\delta = \varepsilon/3$*. Then,* $|x - c| < \delta$ *implies*

$$|f(x) - f(c)| = 3|x - c| < 3\left(\frac{\varepsilon}{3}\right) = \varepsilon.$$

Of particular importance for this discussion is the fact that **the choice of** δ **is the same regardless of which point** $c \in \mathbb{R}$ **we are considering.**

Graph of f

Graph of $f(x) = 3x + 1$



Example 3.

Let's contrast the earlier example with what happens when we prove $g(x) = x^2$ *is continuous on* $\mathbb{R}$.
Given $c \in \mathbb{R}$, *we have*

$$|g(x) - g(c)| = |x^2 - c^2| = |x - c|\ |x + c|.$$

We need to get a bound for $|x + c|$ *that does not depend on* x.
We notice that if $|x - c| < 1$, *say, then* $c - 1 < x < c + 1$, *so* $|x| < |c| + 1$ *and hence* $|x + c| \leq |x| + |c| < 2|c| + 1$.
Thus we have

$$|g(x) - g(c)| < |x - c|\ (2|c| + 1)$$

provided $|x - c| < 1$.

Stephen-116

Example 3 (contd...).

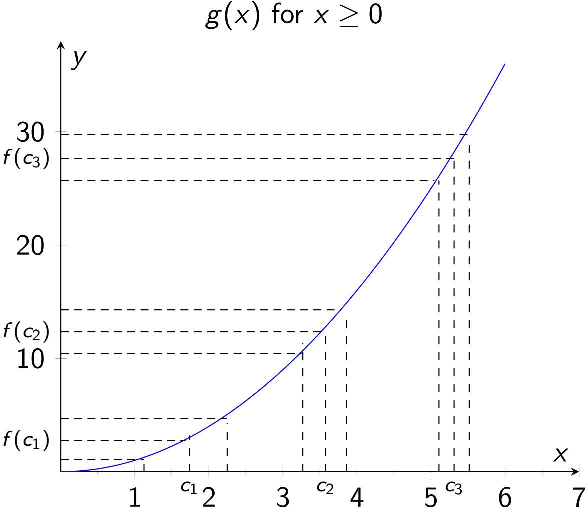
Let $\varepsilon > 0$ *be given.*
To arrange for $|x - c|\ (2|c| + 1) < \varepsilon$, *it suffices to have* $|x - c| < \frac{\varepsilon}{2|c|+1}$ *and also* $|x - c| < 1$. *So we put*

$$\delta(\varepsilon, c) = \min\left\{1, \frac{\varepsilon}{2|c| + 1}\right\}.$$

The work above shows that $|x - c| < \delta$ *implies* $|g(x) - g(c)| < \varepsilon$, *as desired.*
Now, there is nothing deficient about this argument, but it is important to notice that **the algorithm for choosing** δ **depends on the value of** c.

Stephen-116

Let us take $\varepsilon = 1$. Then $\delta = \min\left\{1, \frac{1}{2|c|+1}\right\} = \frac{1}{2|c|+1}$.



Stephen-116

Example 3 (contd...).

From the graph of the function, we may observe that if $|x|$ is very large, small increments in x produce large differences in the values of g taken at these points.

So we **guess** *that the function g does not produce a common $\delta > 0$ on $\mathbb{R}$.*

Stephen-116, Bartle-137

Example 3 (contd...).

The statement

$$\delta(\varepsilon, c) = \min\left\{1, \frac{\varepsilon}{2|c|+1}\right\} \tag{1}$$

means that larger values of c are going to require smaller values of δ.

When c is increasing, the **steepness of the graph** *of $g(x)$ tells that a much smaller δ is required. When $c = 1$, the choice $\delta = 1/3$ is sufficient. But when $c = 10$, a much smaller δ is required.*

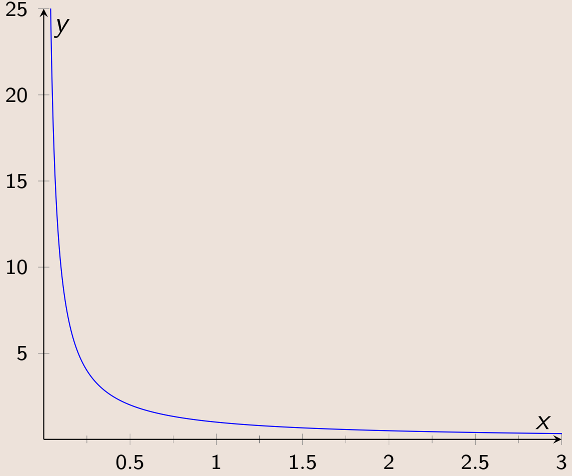
*We note that the value of $\delta(\varepsilon, u)$ given in (1) certainly depends on the point $c \in \mathbb{R}$. If we wish to consider all $c \in \mathbb{R}$, formula (1) does not lead to one value $\delta(\varepsilon) > 0$ that will "***work simultaneously***" for all $c \in \mathbb{R}$, since* $\inf\{\delta(\varepsilon, c) : c \in \mathbb{R}\} = 0$.

Stephen-116, Bartle-137

Another Example

Example 4.

We now show that $h(x) = \frac{1}{x}$ *is continuous on* $(0, \infty)$.



Another Example

Given $c \in (0, \infty)$, we have

$$|h(x) - h(c)| = \left|\frac{1}{x} - \frac{1}{c}\right| = \frac{|x-c|}{x.c} \quad [\text{ since } x > 0, c > 0 \text{ }].$$

We need to get a bound for $\frac{1}{x.c}$ that does not depend on x.

We notice that if $|x - c| < \frac{c}{2}$, say, then $-\frac{c}{2} < x - c < \frac{c}{2}$, so $\frac{c}{2} < x < \frac{3c}{2}$ and hence $\frac{1}{x} < \frac{2}{c}$.

Thus we have

$$|h(x) - h(c)| < (2/c^2) \; |x - c|$$

provided $|x - c| < \frac{c}{2}$.

Another Example

Let $\varepsilon > 0$ be given. To arrange for $(2/c^2)\,|x-c| < \varepsilon$, it suffices to have $|x-c| < \frac{\varepsilon c^2}{2}$ and also $|x-c| < \frac{c}{2}$. So we put

$$\delta(\varepsilon, c) = \min\left\{\frac{c}{2}, \frac{c^2\varepsilon}{2}\right\}. \qquad (1)$$

Thus if $|x-c| < \delta(\varepsilon, c)$, then $|h(x) - h(c)| < (\frac{2}{c^2})(\frac{c^2\varepsilon}{2}) = \varepsilon$.

We have seen that the selection of $\delta(\varepsilon, c)$ by the formula (1) "works" in the sense that it enables us to give a value of δ that will ensure that $|h(x) - h(c)| < \varepsilon$ when $|x-c| < \delta$ and $x, c \in (0, \infty)$.

We note that the value of $\delta(\varepsilon, u)$ given in (1) certainly depends on the point $c \in (0, \infty)$. If we wish to consider all $c \in (0, \infty)$, formula (1) does not lead to one value $\delta(\varepsilon) > 0$ that will "**work**" simultaneously for all $c > 0$, since $\inf\{\delta(\varepsilon, c) : c > 0\} = 0$.

There are other selections that can be made for δ.

There are other selections that can be made for δ. We can start with $|x - c| < \frac{c}{3}$; then $-\frac{c}{3} < x - c < \frac{c}{3}$, so $\frac{2c}{3} < x < \frac{4c}{3}$ and hence $\frac{1}{x} < \frac{3}{2c}$.

Thus we have
$$|h(x) - h(c)| < (3/2c^2)\ |x - c|$$
provided $|x - c| < \frac{c}{3}$.

Therefore we could take
$$\delta_1(\varepsilon, c) = \min\left\{\frac{c}{3}, \frac{2c^2\varepsilon}{3}\right\}.$$

However, we still have $\inf\{\delta_1(\varepsilon, c) : c > 0\} = 0$.

In fact, there is no way of choosing one value of δ that will "**work**" for all $c > 0$ for the function $h(x) = 1/x$, as we shall see.

Bartle-137

As $c \to 0$, we see that $h(c)$ goes to 'infinity'. Thus, it is intuitively clear that if we want to control the value of $h(x)$ for x near to c, nearer c to 0, smaller value for δ is required.

In other words, if we want to assert that $h(x)$ is within ε-distance of $h(c)$, then we may have to restrict x to smaller and smaller open intervals around c as c goes nearer and nearer to 0.

We have analyzed the behaviour of δ required to prove the continuity of f at c varying in $(0, \infty)$. It is observed that given $\varepsilon > 0$, we cannot make a single choice of δ which will work for all c in $(0, \infty)$, or, even in $(0, 1)$.

Kumaresan-75

Motivation

Motivated by Examples 3 and 4, it turns out to be very useful to know when the δ in the definition (of continuity) can be chosen to depend only on $\varepsilon > 0$ and A, so that δ does not depend on the particular point c.

We shall give a special name for functions satisfying the above condition by giving the following definition. Such functions are said to be uniformly continuous on A.

Definition 5.

A function $f : X \to Y$ is said to be **uniformly continuous on** $A \subseteq X$ *if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $x, y \in A$,*

$$d(x,y) < \delta \quad \textit{implies} \quad \rho(f(x), f(y)) < \varepsilon.$$

Karunakaran-5-25

Note that while the continuity of f at a point is discussed, we are concerned only with the values (behaviour) of the function near the point under consideration.

But when we wish to say that a function is uniformly continuous on its domain, we need to know the values of f on the entire domain. Hence the continuity is known as a local concept whereas the uniform continuity is known as a global concept.

Uniform continuity is defined on a set; unlike continuity, it has no local counterpart. The best way to understand uniform continuity is to look at some examples and see how the behaviour of the function on the entire domain plays a role.

Kumaresan-76, Michael-147

Uniform Continuity

Uniform continuity is a property concerning a function and a set [on which it is defined]. It makes no sense to speak of a function being uniformly continuous at a point.

Uniform continuity is always discussed in reference to a particular domain.

The criterion for uniform continuity is very like the epsilon-delta criterion for continuity, but, in this case, for each $\varepsilon > 0$, there is a $\delta > 0$ that serves the purpose of the definition right across the subset.

Kumaresan-76, Ross-134

Caution !

Example 6.

Consider $f(x) = \sqrt{x}$ *defined on* $A = \{x \in \mathbb{R} : x \geq 0\}$. *We give an* ε-δ *proof for the fact that* f *is continuous on* A.

Let $\varepsilon > 0$. *We need to argue that* $|f(x) - f(c)|$ *can be made less than* ε *for all values of* x *in some* δ *neighborhood around* c.

If $c = 0$, *this reduces to the statement* $\sqrt{x} < \varepsilon$, *which happens as long as* $x < \varepsilon^2$. *Thus, if we choose* $\delta = \varepsilon^2$, *we see that* $\sqrt{x} < \varepsilon$, *which happens as long as* $x < \varepsilon^2$.

Thus, if we choose $\delta = \varepsilon^2$, *we see that* $|x - 0| < \delta$ *implies* $|f(x) - 0| < \varepsilon$.

Stephen-112

Example 6 (contd...).

For a point $c \in A$ different from zero, we need to estimate $|\sqrt{x} - \sqrt{c}|$. This time, write

$$|\sqrt{x} - \sqrt{c}| = |\sqrt{x} - \sqrt{c}| \left(\frac{\sqrt{x} + \sqrt{c}}{\sqrt{x} + \sqrt{c}} \right) = \frac{|x - c|}{\sqrt{x} + \sqrt{c}} \leq \frac{|x - c|}{\sqrt{c}}.$$

In order to make this quantity less than ε, it suffices to pick $\delta = \varepsilon\sqrt{c}$.

Then, $|x - c| < \delta$ implies $|\sqrt{x} - \sqrt{c}| < \frac{\varepsilon\sqrt{c}}{\sqrt{c}} = \varepsilon$, as desired.

The above example is included to keep us from jumping to the **erroneous conclusion** that functions that are having δ depending of the point c chosen in the set A, are not uniformly continuous. We shall prove later that $f(x) = \sqrt{x}$ is uniformly continuous on $A = \{x \in \mathbb{R} : x \geq 0\}$.

Theorem 7.

Let X and Y be metric spaces. Let $f : X \to Y$ and $A \subseteq X$.

(i) *If f is uniformly continuous on A, then f is continuous on A.*

(ii) *If f is uniformly continuous on X, then f is uniformly continuous on A.*

Uniform continuity is a strictly stronger property.

We shall see several examples of continuous functions which are not necessarily uniformly continuous. **Continuous functions that are not uniformly continuous abound.**

Michael-147

The diameter of A is the extended real number defined by

$$diam(A) := \begin{cases} \sup\{d(x,y) : x, y \in A\} & \text{if this quantity is finite} \\ +\infty & \text{otherwise.} \end{cases}$$

In the finite case, this is, by the definition of the supremum, the smallest real number D such that any two points of A are at most a distance D apart.

Exercise 8.

Let $f : X \to Y$ be a map of metric spaces. Then f is uniformly continuous on A if and only if for every $\varepsilon > 0$, there is a $\delta > 0$ such that whenever $E \subseteq A$ with $diam(E) < \delta$, we have $diam(f(E)) < \varepsilon$.

Criterion for Non-Uniform Continuity

It is useful to formulate a condition equivalent to saying that f is not uniformly continuous on A. We give such criterion in the next result.

Theorem 9 (criterion for non-uniform continuity).

Let X and Y be metric spaces. Let $f : X \to Y$ and A be a non-empty subset of X. The following statements are equivalent :

1. *f is* **not uniformly continuous on** *A ;*
2. *there exists an $\varepsilon_0 > 0$ such that for every $\delta > 0$, no matter how small, it is possible to find $x, y \in A$ (depending on δ) with $d(x, y) < \delta$ but $\rho(f(x), f(y)) \geq \varepsilon_0$;*

Non-Uniform Continuity - A Sequential Characterization

Theorem 10 (sequential criterion for non-uniform continuity).

Let X and Y be metric spaces and $f : X \to Y$. Then f is not uniformly continuous on A if and only if there exist an $\varepsilon_0 > 0$ and two sequences (x_n) and (y_n) in A satisfying

$$d(x_n, y_n) \to 0 \quad \textit{but} \quad \rho(f(x_n), f(y_n)) \geq \varepsilon_0 \quad \textit{for all } n \in \mathbb{N}.$$

Proof : Apply Theorem 9 for $\delta = 1/n$, for every $n \in \mathbb{N}$ to get the sequences (x_n) and (y_n) in A.

Theorem 11 (sequential criterion for uniform continuity).

Let X and Y be metric spaces and $f : X \to Y$. Then f is uniformly continuous on $A \subseteq X$ iff for every sequences (x_n) and (y_n) in A satisfying $d(x_n, y_n) \to 0$, we have $\rho(f(x_n), f(y_n)) \to 0$.

Proof. $\Rightarrow)$: Suppose $f : X \to Y$ is uniformly continuous on $A \subseteq X$. Let (x_n) and (y_n) be sequences in A such that $d(x_n, y_n) \to 0$. Let $\varepsilon > 0$ be given.

Since f is uniformly continuous on A, there exists $\delta > 0$ such that $x, y \in A$ with

$$d(x, y) < \delta \quad \text{implies} \quad \rho(f(x), f(y)) < \varepsilon.$$

As $d(x_n, y_n) \to 0$, there exists a natural number $N_0 \in \mathbb{N}$ be such that $d(x_n, y_n) < \delta$ for all $n \geq N_0$. Then, it follows that $\rho(f(x_n), f(y_n)) < \varepsilon$ for all $n \geq N_0$.

Proof (contd...)

$\Leftarrow$) : Suppose f is not uniformly continuous. By Theorem 9 (criterion for non-uniform continuity), there exists $\varepsilon_0 > 0$ such that for every $\delta > 0$, there exist $x, y \in A$ (depending on δ) such that

$$d(x, y) < \delta, \quad \text{but} \quad \rho(f(x), f(y)) \geq \varepsilon_0.$$

In particular, for every $n \in \mathbb{N}$, there exist $x_n, y_n \in A$ such that $d(x_n, y_n) < \frac{1}{n}$, but $\rho(f(x_n), f(y_n)) \geq \varepsilon_0$.

Thus, we have proved that, if f is not uniformly continuous, then there exist sequences (x_n) and (y_n) in A satisfying

$$d(x_n, y_n) \to 0, \quad \text{but} \quad \rho(f(x_n), f(y_n)) \not\to 0.$$

This completes the proof.

Example - Uniformly Continuous

Example 12.

We have noticed (in Example 2) that the function $f(x) = 3x + 1$ is continuous on $\mathbb{R}$.

Since

$$|f(x) - f(y)| = |(3x + 1) - (3y + 1)| = 3|x - y|,$$

so, given $\varepsilon > 0$, we choose $\delta = \varepsilon/3$. Then, $|x - c| < \delta$ implies

$$|f(x) - f(c)| = 3|x - c| < 3\left(\frac{\varepsilon}{3}\right) = \varepsilon.$$

Hence the function $f(x) = 3x + 1$ is uniformly continuous on $\mathbb{R}$.

Example 13.

We have observed (in Example 3) that the function $g(x) = x^2$ could not be uniformly continuous on $\mathbb{R}$ because larger values of x require smaller and smaller values of δ.

By applying Theorem 10, for $\varepsilon_0 = 2$, $x_n = n$ and $y_n = n + 1/n$, the function $g(x) = x^2$ is not uniformly continuous on $\mathbb{R}$ because

$$|x_n - y_n| = \frac{1}{n} \to 0$$

and

$$|g(x_n) - g(y_n)| = \left| n^2 - \left(n + \frac{1}{n}\right)^2 \right| = 2 + \frac{1}{n^2} \geq 2 \quad \text{for all } n \in \mathbb{N}.$$

Stephen-118

Example 14.

However, that $g(x)$ is uniformly continuous on the bounded set $[-10, 10]$. Notice that if we restrict our attention to the domain $[-10, 10]$ then $|x + y| \leq 20$ for all x and y.

Given $\varepsilon > 0$, we can now choose $\delta = \varepsilon/20$, and verify that if $x, y \in [-10, 10]$ satisfy $|x - y| < \delta$, then

$$|g(x) - g(y)| = |x^2 - y^2| = |x - y||x + y| < \left(\frac{\varepsilon}{20}\right) 20 = \varepsilon.$$

In fact, it is not difficult to see how to modify this argument to show that $g(x)$ is uniformly continuous on any bounded subset A of $\mathbb{R}$.

Stephen-118

Example 15.

We have observed (in Example 4) that the function $h(x) = \frac{1}{x}$ *is continuous on* $(0, \infty)$*. By applying Theorem 10, for* $\varepsilon_0 = 1$*,* $x_n = \frac{1}{\sqrt{n+2}}$ *and* $y_n = \frac{1}{n+2}$*,*

the function $h(x) = \frac{1}{x}$ *is not uniformly continuous on* $(0, \infty)$ *because*

$$|x_n - y_n| = \left|\frac{1}{\sqrt{n+2}} - \frac{1}{n+2}\right| \to 0$$

and

$$|h(x_n) - h(y_n)| = \left|\sqrt{n+2} - (n+2)\right| \geq 1 \quad \text{for all } n \in \mathbb{N}.$$

However, that $h(x)$ *is uniformly continuous on any set of the form* $[a, \infty)$*, where* a *is a positive constant.*

Examples - Uniformly Continuous

Example 15 (contd...).

Let $\varepsilon > 0$. *We need to show that there exists* $\delta > 0$ *such that*

$$x \geq a,\ y \geq a \text{ and } |x - y| < \delta \quad \text{imply} \quad |h(x) - h(y)| < \varepsilon. \tag{1}$$

We have

$$|h(x) - h(y)| = \left|\frac{1}{x} - \frac{1}{y}\right| = \frac{|x - y|}{|x|\ |y|} \leq \frac{|x - y|}{a^2}.$$

So we set $\delta = \varepsilon a^2$. *It is now straightforward to verify* (1).

Example 16.

We now show that $h(x) = \frac{1}{x^2}$ is uniformly continuous on any set of the form $[a, \infty)$ where a is a positive constant.

Let $\varepsilon > 0$. We need to show that there exists $\delta > 0$ such that
$$x \geq a,\ y \geq a \text{ and } |x - y| < \delta \quad \text{imply} \quad |h(x) - h(y)| < \varepsilon. \tag{1}$$
We have
$$h(x) - h(y) = \frac{(y - x)(y + x)}{x^2 y^2}$$
If we can show that $\frac{y+x}{x^2y^2}$ is bounded on $[a, \infty)$ by a constant M, then we will take $\delta = \frac{\varepsilon}{M}$.

Ross-134

Example 16 (contd...).

But we have

$$\frac{y+x}{x^2y^2} = \frac{1}{x^2y} + \frac{1}{xy^2} \leq \frac{1}{a^3} + \frac{1}{a^3} = \frac{2}{a^3},$$

so we set $\delta = \frac{\varepsilon a^3}{2}$. *It is now straightforward to verify* (1). *In fact,* $x \geq a, y \geq a$ *and* $|x-y| < \delta$ *imply*

$$|h(x) - h(y)| = \frac{|y-x| \cdot |y+x|}{x^2y^2} < \frac{2\delta}{a^3} = \varepsilon.$$

We have shown that $h(x) = \frac{1}{x^2}$ *is uniformly continuous on* $[a, \infty)$ *(where* $a > 0$ *is fixed) since* δ *depends only on* ε *and the set* $[a, \infty)$.

Kumaresan-76

Using ε-δ, we prove non-uniform continuity of $g(x) = x^2$ on the set $[0, \infty)$.

Using the ε-δ definition (criterion for non-uniform continuity), we now prove that the function $g(x) = x^2$ (in Example 3) is not uniformly continuous on the set $[0, \infty)$.

We can choose $\varepsilon_0 = 1$ and then for any $\delta > 0$, we have

$$|g(x + \delta) - g(x)| = (x + \delta)^2 - x^2 = 2x\delta + \delta^2$$

and we can choose $x > 1/(2\delta)$ so that $2x\delta > 1$.

Hence $|g(x + \delta) - g(x)| \geq 1$.

So there is no δ that works for every x in the infinite interval.

Thus $g(x) = x^2$ is not uniformly continuous on $[0, \infty)$.

Using ε-δ, we prove non-uniform continuity of $h(x) = \frac{1}{x^2}$ on $(0, \infty)$ or even on the set $(0, 1)$.

Using the ε-δ definition (criterion for non-uniform continuity), we now prove that the function $h(x) = \frac{1}{x^2}$ (in Example 4) is not uniformly continuous on the set $(0, \infty)$ or even on the set $(0, 1)$.

We will prove this by directly violating the definition of uniform continuity for $\varepsilon_0 = 1$. That is, for each $\delta > 0$ there exist x, y in $(0, 1)$ such that

$$|x - y| < \delta \quad \text{and yet} \quad |h(x) - h(y)| \geq 1. \tag{1}$$

To show (1) it suffices to take $y = x + \frac{\delta}{2}$ and arrange for

$$\left| h(x) - h\left(x + \frac{\delta}{2}\right) \right| \geq 1. \tag{2}$$

The motivation for this calculation is to go from two unknowns, x and y, in (1) to one unknown, x, in (2).

Ross-135

Using ε-δ, we prove non-uniform continuity of $h(x) = \frac{1}{x^2}$ on $(0, \infty)$ or even on the set $(0, 1)$.

(2) is equivalent to

$$1 \leq \frac{(x + \frac{\delta}{2} - x)(x + \frac{\delta}{2} + x)}{x^2(x + \frac{\delta}{2})^2} = \frac{\delta(2x + \frac{\delta}{2})}{2x^2(x + \frac{\delta}{2})^2}. \tag{3}$$

It suffices to prove (1) for $\delta < \frac{1}{2}$. To obtain (3), let us try $x = \delta$. Then

$$\frac{\delta(2\delta + \frac{\delta}{2})}{2\delta^2(\delta + \frac{\delta}{2})^2} = \frac{\frac{5\delta^2}{2}}{\frac{9\delta^4}{2}} = \frac{5}{9\delta^2} \geq \frac{5}{9(\frac{1}{2})^2} = \frac{20}{9} > 1.$$

To summarize, we have shown that if $0 < \delta < \frac{1}{2}$, then $|h(\delta) - h(\delta + \frac{\delta}{2})| > 1$, so (1) holds with $x = \delta$ and $y = \delta + \frac{\delta}{2}$.

Thus $h(x) = \frac{1}{x^2}$ is not uniformly continuous on $(0, 1)$.

Example of uniformly continuous functions

We shall now give some examples of uniformly continuous functions,

Example 17.

Consider $f : [a, b] \to \mathbb{R}$ *defined by* $f(x) = \frac{k}{x-2}$ *where* a, b, k *are positive constants with* $a > 2$. *We now show that* f *is uniformly continuous on* $[a, b]$. *Let* $\varepsilon > 0$ *be given. Now*

$$f(x) - f(y) = \frac{k}{x-2} - \frac{k}{y-2} = \frac{k(y-x)}{(x-2)(y-2)}$$

Since $a > 2$, $a = 2 + \eta$ *for some* $\eta > 0$ *and if* $x, y \in [a, b]$, *then* $|x-2| = x - 2 \geq \eta$ *and* $y - 2 \geq \eta$. *Choose* $\delta < \frac{\eta^2 \varepsilon}{k}$. *Thus if* $|x - y| < \delta$ *and* $x, y \in [a, b]$, *then*

$$|f(x) - f(y)| = \frac{k|x-y|}{(x-2)(y-2)} \leq \frac{k\delta}{\eta^2} < \varepsilon.$$

Karunakaran 5-26

Example of uniformly continuous functions

Example 18.

Let $f : (0,6) \to \mathbb{R}$ *defined by* $f(x) = x^2 + 2x - 5$. *We now show that* f *is uniformly continuous on* $(0,6)$. *For given* $\varepsilon > 0$, *choose* $\delta = \frac{\varepsilon}{14}$. *Now for any* $x, y \in (0,6)$ *with* $|x - y| < \delta$, *we have*

$$|f(x) - f(y)| = |x - y||x + y + 2| < 14\delta < \varepsilon$$

(since $0 \leq x, y < 6$, *we have* $|x + y + 2| < 14$*).*

Karunakaran 5-26

Example 19.

The exponential function $x \mapsto e^x$ *defined on* $\mathbb{R}$ *is not uniformly continuous on* $\mathbb{R}$*. Indeed, there is no* $\delta > 0$ *that guarantees*

$$|e^x - e^y| < 1 \quad \text{for all} \;\; x, y \in \mathbb{R} \quad \text{with} \;\; |x - y| < \delta.$$

Specifically, for all $x, y \in \mathbb{R}$ *with* $x < y$*, we have*

$$(e^y - e^x)/(y - x) > e^x.$$

Then, for any $\gamma \in (0, 1)$*, pick* $x = -\ln \gamma$ *and* $y = \gamma - \ln \gamma$*, so that*

$$e^y - e^x > (y - x)e^x = 1.$$

Michael-148

Continuous functions that are not uniformly continuous abound.

Exercises 20.

Prove the following statements :

1. *No polynomial function of degree greater than* 1 *is uniformly continuous on* $\mathbb{R}$.
2. *The logarithmic function is not uniformly continuous on* $(0, \infty)$.

Example - Continuous on Bounded Domain - Not Uniformly Continuous

The following example is included to keep us from jumping to the **erroneous conclusion** that functions that are continuous on bounded domains are necessarily uniformly continuous.

Example 21.

Consider $h : (0,1] \to \mathbb{R}$ defined by $h(x) = \frac{1}{x^2}$. This function is continuous on $(0,1]$, but not uniformly continuous on $(0,1]$. We apply Theorem 10.

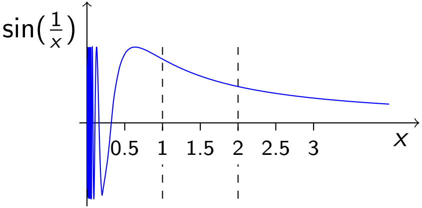
To see this, take $\varepsilon_0 = 2$ and consider the sequences $(\frac{1}{\sqrt{n}})$ and $(\frac{1}{n})$.

We see that $\left|\frac{1}{\sqrt{n}} - \frac{1}{n}\right| \to 0$, but $\left|h(\frac{1}{\sqrt{n}}) - h(\frac{1}{n})\right| = |n - n^2| \geq 1$, for all $n \in \mathbb{N}$.

Example - Continuous Bounded Function - Not Uniformly Continuous

Example 22.

$f(x) = \sin(\frac{1}{x})$ is a bounded continuous function on $(0, 1)$ but it is not uniformly continuous on $(0, 1)$.



The function $f(x) := \sin(\frac{1}{x})$ (for $x > 0$) changes its values rapidly near "zero".

Example - Not Uniformly Continuous

Example 15 (contd...).

The problem arises near zero, where the increasingly rapid oscillations take domain values that are quite close together to range values a distance 2 apart. To illustrate Theorem 10, take $\varepsilon_0 = 2$ and set

$$x_n = \frac{1}{\pi/2 + 2n\pi} \quad \text{and} \quad y_n = \frac{1}{3\pi/2 + 2n\pi}.$$

Because each of these sequences tends to zero, we have $|x_n - y_n| \to 0$, and a short calculation reveals $|f(x_n) - f(y_n)| = 2$ for all $n \in \mathbb{N}$.

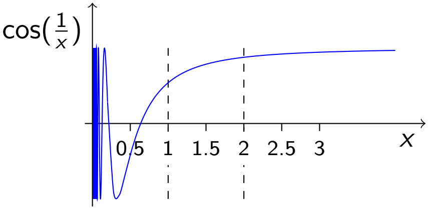
The above example reveals that bounded continuous functions can fail to be uniformly continuous if they oscilate arbitrarily quickly.

Stephen-117

Example - Not Uniformly Continuous

Exercise 23.

$g(x) = \cos(\frac{1}{x})$ is a bounded continuous function on $(0,1)$ but it is not uniformly continuous on $(0,1)$.



Continuous functions that are not uniformly continuous abound.

Every uniformly continuous function on A is continuous on A. We have seen some examples of functions which are not uniformly continuous.

- Continuous function defined on a bounded set which is not uniformly continuous. (Example 21);
- Continuous bounded function which is not uniformly continuous. (Example 22 and 23).

Continuity and uniform continuity are the same on compact sets.

The importance of uniform continuity lies not so much in knowing which functions are uniformly continuous as in knowing on which sets continuity of a given function is uniform. The most useful theorem in this regard is that every continuous function is uniformly continuous on all compact subsets of its domain.

We shall prove that continuous on compact domains are necessarily uniformly continuous. So, when the domain is compact, uniform continuity and continuity are the same on the domain.

Michael-148

Compact Sets

Definition 24.

A subset K of a metric space (X, d) is said to be **compact** *if every sequence in K has a convergent subsequence that converges to a limit in K.*

Definition 25 (Equivalent definition).

By an **open cover** *of a set A in a metric space X we mean a collection $\{G_\alpha\}$ of open subsets of X such that $E \subseteq U_\alpha G_\alpha$.*
A subset K of a metric space (X, d) is said to be **compact** *if every open cover of K contains a finite subcover.*

- Every finite set is compact.
- Compact subsets of metric spaces are closed.
- Closed subsets of compact sets are compact.
- The Heine-Borel theorem states that a subset of $\mathbb{R}^n$ (with the usual topology) is compact iff it is closed and bounded.

Eevery continuous function is uniformly continuous on all compact subsets of its domain.

Theorem 26 (Uniform Continuity Theorem).

Let X and Y be metric spaces and let K be a compact subset of X. If $f : X \to Y$ is continuous on K, then f is uniformly continuous on K.

Proof : Suppose f is not uniformly continuous on K. Then, by Theorem 10, there exists an $\varepsilon_0 > 0$ and two sequences (x_n) and (y_n) in K satisfying

$$d(x_n, y_n) \to 0 \quad \text{but} \quad \rho(f(x_n), f(y_n)) \geq \varepsilon_0 \quad \text{for all } n \in \mathbb{N}. \qquad (\star).$$

Since K is compact, (x_n) and (y_n) have convergent subsequences, say $(\tilde{x}_n)$ and $(\tilde{y}_n)$, converging to x and y in K, respectively. That is, $d(\tilde{x}_n, x) \to 0$ and $d(\tilde{y}_n, y) \to 0$.

Also we have $d(\tilde{x}_n, \tilde{y}_n) \to 0$.

Ross-161

Proof (contd...)

By $\varepsilon/3$-argument, for any $\varepsilon > 0$, there are natural numbers N_1, N_2, N_3 such that

$$d(\tilde{x}_n, x) < \frac{\varepsilon}{3}, \text{ for all } n \geq N_1 \; ; \; d(\tilde{y}_n, y) < \frac{\varepsilon}{3}, \text{ for all } n \geq N_2 \; ; \; d(\tilde{x}_n, \tilde{y}_n) < \frac{\varepsilon}{3}, \text{ for all } n \geq N_3.$$

Let $N_0 = \max\{N_1, N_2, N_3\}$. Thus

$$d(x, y) \leq d(x, x_n) + d(x_n, y_n) + d(y_n, y) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon, \quad \text{for all } n \geq N_0.$$

Since the left-hand side is independent of n, $d(x, y) < \varepsilon$, for every $\varepsilon > 0$. Thus $x = y$.

Since both $(\tilde{x}_n)$ and $(\tilde{y}_n)$ converge to x and f is continous on K, $\rho(f(\tilde{x}_n), f(\tilde{y}_n)) \to 0$, this is a contradiction to $(\star)$.

Ross-161

Another Proof

Proof : Let $\varepsilon > 0$ be given.

Since f is continuous on K, for every $u \in K$, there exists $\delta_u > 0$ such that $x \in K$ and

$$d(x,u) < \delta_u \quad \text{implies} \quad \rho(f(x), f(u)) < \frac{\varepsilon}{2}. \tag{1}$$

Since $\{B_X\left(u, \frac{\delta_u}{2}\right) : u \in K\}$ is an open cover of K and K is compact, there exist $u_1, \ldots, u_n$ in K such that

$$K \subseteq U_{i=1}^n B_X(u_i, \frac{\delta_{u_i}}{2}). \tag{2}$$

Let

$$\delta = \frac{1}{2} \min\left\{\delta_{u_1}, \delta_{u_2}, \ldots, \delta_{u_n}\right\}.$$

Ross-161

Another Proof (contd...)

Now, let $x, y \in K$ be such that $d(x, y) < \delta$.
Since $x \in K$, by (2), $x \in B_X(u_i, \frac{\delta_{u_i}}{2})$ for some $i \in \{1, 2, \ldots, n\}$, so $d(x, u_i) < \frac{\delta_{u_i}}{2}$. Then we have

$$d(y, u_i) \leq d(y, x) + d(x, u_i) < \delta + \frac{\delta_{u_i}}{2} < \frac{\delta_{u_i}}{2} + \frac{\delta_{u_i}}{2} = \delta_{u_i}.$$

Therefore, applying (1) twice, we have

$$\rho(f(x), f(u_i)) < \frac{\varepsilon}{2} \quad \text{and} \quad \rho(f(y), f(u_i)) < \frac{\varepsilon}{2}.$$

Hence,

$$\rho(f(x), f(y)) \leq \rho(f(x), f(u_i)) + \rho(f(u_i), f(y)) < \varepsilon$$

as desired. This completes the proof. Ross-161

An alternatre proof is sketched in the book by Rudin (page-99).

Compactness is essential.

Theorem 27.

Let E be a subset of $\mathbb{R}$. If E is bounded but not closed, then there exists a continuous function on E which is not uniformly continuous on E.

Proof : Since E is bounded but not closed, there exists a limit point a of E which is not a point of E. Consider

$$f(x) = \frac{1}{x-a} \quad \text{for } x \in E. \qquad (1)$$

Obviously, f is continuous on E (being a quotient of two continuous functions with denominator non-zero).

Since a is a limit point of E, given any real number $M > 0$ there exists $x_0 \in E$ such that $|x_0 - a| < \frac{1}{M}$ or that $|f(x_0)| > M$. This shows that f is unbounded on E.

Rudin-92, Karunakaran-5-27

Proof (contd...)

To see that (1) is not uniformly continuous, let $\varepsilon > 0$ and $\delta > 0$ be arbitrary, and choose a point $x \in E$ such that $|x - x_0| < \delta$. Taking y close enough to x_0, we can then make the difference $|f(x) - f(y)|$ greater than ε, although $|x - y| < \delta$. Since this is true for every $\delta > 0$, f is not uniformly continuous on E.

Indeed, using the definitions, for each $\varepsilon > 0$ and each $\delta > 0$ we can choose points $p, q \in S$ such that

$$|p - a| < \delta \quad \text{and} \quad |q - a| < \min\{\delta - |p - a|, \varepsilon + \frac{1}{|p - a|}\}.$$

Now

$$|p - q| \leq |p - a| + |q - a| < \delta$$

but

$$\begin{aligned}
|f(p) - f(q)| &= \left| \frac{1}{p - a} - \frac{1}{q - a} \right| \\
&= \left| \frac{p - q}{(p - a)(q - a)} \right| \geq \frac{|p - a| - |q - a|}{|(p - a)(q - a)|} \\
&= \frac{1}{|q - a|} - \frac{1}{|p - a|} \\
&\geq \varepsilon + \frac{1}{|p - a|} - \frac{1}{|p - a|} = \varepsilon
\end{aligned}$$

Hence f is not uniformly continuous.

Rudin-92, Karunakaran-5-28

Boundedness is essential.

We may also note that if E is unbounded, it is possible to have a continuous function, which is uniformly continuous.

Theorem 27 would be false if boundedness were omitted from the hypothesis. There are unbounded subsets of $\mathbb{R}$ on which every function is uniformly continuous. The following example illustrates that boundedness of E alone is not sufficient to guarantee the existence of non-uniform continuous function.

Example 28.

Let $\mathbb{Z}$ be the set of all integers. Then every function $f : \mathbb{Z} \to \mathbb{R}$ is not only continuous but is uniformly continuous on $\mathbb{Z}$.

Take any $\varepsilon > 0$. Then choose $\delta < 1$. Then we prove $|x - y| < \delta \implies |f(x) - f(y)| < \varepsilon$. But $|x - y| < \delta$ means, there is only one point in $(y - \delta, y + \delta)$, so $f(x) = f(y)$ and hence the result follows.

Rudin-92, Karunakaran-5-29

Example 29.

The sine and cosine functions are uniformly continuous on the whole of $\mathbb{R}$*. To see this, Theorem 26 ensures that they are both uniformly continuous on the interval* $[-\pi, \pi]$*.*

Then 2π*-periodicity of the functions clinches the matter: let* $\varepsilon > 0$ *and let* $\delta > 0$ *be such that, for all* $a, b \in [-\pi, \pi]$ *with* $|b - a| < \delta$*, we have* $|\sin b - \sin a| < \varepsilon$*.*

Then for all $x, y \in \mathbb{R}$*, there exist* $a, b \in [-\pi, \pi]$ *with* $|a - b| \leq |x - y|$ *and* $\sin x = \sin a$ *and* $\sin y = \sin b$*, from which the result follows easily.*

Michael-149

Exponential Function and All Polynomial Functions

Example 30.

In view of Theorem 26, the following functions are uniformly continuous on the indicated sets:

- x^{73} *on* $[-13, 13]$ *;*
- $\sqrt{x}$ *on* $[0, 400]$ *;*
- $x^{17}\sin(e^x) - e^{4x}\cos 2x$ *on* $[-8\pi, 8\pi]$ *;*
- $\frac{1}{x^6}$ *on* $[\frac{1}{4}, 44]$*, and so on.*

Exercise 31.

The exponential function and all polynomial functions, being continuous on $\mathbb{R}$*, are uniformly continuous on every bounded subset of* $\mathbb{R}$*.*

Michael-149, Ross-137

Example 32.

The tangent function is continuous but not uniformly continuous on the bounded interval $(-\pi/2, \pi/2)$ *of* $\mathbb{R}$.

Theorem 26 does not apply here because the function is not continuous on any superset of $(-\pi/2, \pi/2)$ *which is compact.*

For $r \in (0, \pi/2)$, *the tangent function is uniformly continuous on the interval* $(-(\pi/2) + r, (\pi/2) - r)$ *because it is continuous on the closed interval* $[-(\pi/2) + r, (\pi/2) - r]$.

Michael-150

Example 33.

Suppose X and Y are metric spaces and $f : X \to Y$. If X is a discrete metric space (4.3.7), then f is continuous irrespective of the metric on Y. This is so because every subset of a discrete metric space is open, so, for each subset V of Y, $f^{-1}(V)$ is necessarily open in X. In particular, if $\mathbb{N}$ is endowed with its usual metric inherited from $\mathbb{R}$, or with the discrete metric, or indeed with the inverse metric $(m, n) \mapsto |m^{-1} - n^{-1}|$, then every function from $\mathbb{N}$ into a metric space is continuous. In other words, all sequences are continuous functions provided $\mathbb{N}$ is endowed with a suitable metric.

There are metric spaces with metrics that may differ from the discrete metric yet generate the same topology, namely the power set. Such spaces are collectively called discrete metric spaces. *Michael-131*

Discrete Metric Spaces

Definition 34.

A metric space (X, d) is called a discrete metric space if its subsets are open (and therefore also closed) in X.

Example 35.

Every finite metric space is a discrete space. $\mathbb{N}$ with its usual metric inherited from $\mathbb{R}$ is a discrete metric space. $\mathbb{N}$ with the metric $(m, n) \mapsto |m^{-1} - n^{-1}|$ is a discrete metric space

Exercise 36.

Show that $\mathbb{Q}$ is a countable metric space that is not a discrete metric space.

Uniformly Continuous Functions

Suppose (X, d) and (Y, ρ) are metric spaces and $f : X \to Y$. If d is the discrete metric, then f is uniformly continuous irrespective of the metric on Y. Specifically, for each $\varepsilon > 0$, we can choose $\delta = 1$ so that whenever $x, y \in X$ with $d(x, y) < 1$. We get that $x = y$, hence $0 = \rho(f(x), f(y)) < \varepsilon$.

It is important in this example that the metric is discrete and the space not merely a discrete space. There are continuous functions on discrete metric spaces that are not uniformly continuous. The subspace $S = \{1/n : n \in \mathbb{N}\}$ of $\mathbb{R}$ is a discrete metric space because each of its singleton sets is both open and closed in S. The function $1/n \mapsto n$ is continuous, as it must be (by Example 33), because S is discrete, but it is not uniformly continuous. Specifically, if $\delta > 0$ and $m, n \in \mathbb{N}$ with $m > n > 2/\delta$, then $|1/m - 1/n| < \delta$ but $|m - n| \geq 1$.

Michael-150

Exercise 37.

Suppose (X, d) *and* (Y, ρ) *are metric spaces,* $f, g : X \to Y$. *If* f *and* g *are uniformly continuous on* $A \subseteq X$, *then* $f + g$ *is uniformly continuous on* A.

Exercise 38 (Composition of uniformly continuous functions are uniformly continuous).

Suppose $(X, d), (Y, \rho)$ *and* (Z, γ) *are metric spaces,* $f : X \to Y$ *and* $g : Y \to Z$. *If* f *and* g *are uniformly continuous on* X *and* $f(X)$, *respectively, then* $g \circ f$ *is uniformly continuous on* X.

Michael-157

Exercise 39 (Product of uniformly continuous functions).

1. *Show that if f and g are uniformly continuous on $A \subseteq \mathbb{R}$ and if they are both bounded on A, then their product fg is uniformly continuous on A.*
   [Hint : If M is a bound for both f and g on A, show that $|f(x)g(x) - f(u)g(u)| \leq M|f(x) - f(u)| + M|g(x) - g(u)|$ for all $x, u \in A$.]
2. *If $f(x) := x$ and $g(x) := \sin x$, show that both f and g are uniformly continuous on $\mathbb{R}$, but that their product fg is not uniformly continuous on $\mathbb{R}$.*

Bartle-144

Exercises 40.

1. *Show that* $f : [1, \infty) \to \mathbb{R}$ *given by* $f(x) = 1/x$ *is uniformly continuous on* $[1, \infty)$.
2. *Show that the function* $f : [1, \infty) \to \mathbb{R}$ *given by* $f(x) = 1/x^n$, *for* $n \in \mathbb{N}$, *is uniformly continuous on* $[1, \infty)$.
3. *Show that* $f : (0, 1) \to \mathbb{R}$ *given by* $g(x) = 1/x$ *cannot be uniformly continuous on* $(0, 1)$, *using*
   (a) ε-δ *criterion for non-uniform continuity ;*
   (b) *sequential criterion for non-uniform continuity.*

Kumaresan-77

Exercises 41.

1. *Show that the function* $f(x) = 1/x$ *is uniformly continuous on the set* $[a, \infty)$*, where* a *is a positive constant.*
   [Hint : Since $1/x - 1/u = (u - x)/xu$*, it follows that* $[1/x - 1/u] \leq (1/a^2)|x - u|$ *for* $x, u \in [a, \infty)$*.]*
2. *Show that the function* $f(x) := 1/(1 + x^2)$ *for* $x \in \mathbb{R}$ *is uniformly continuous on* $\mathbb{R}$*.*
3. *Prove that if* f *and* g *are each uniformly continuous on* $\mathbb{R}$*, then the composite function* $f \circ g$ *is uniformly continuous on* $\mathbb{R}$*.*
   [Hint : Given $\varepsilon > 0$ *there exists* $\delta_f > 0$ *such that* $|y - v| < \delta_f$ *implies* $|f(y) - f(v)| < \varepsilon$*. Now choose* $\delta_g > 0$ *so that* $|x - u| < \delta_g$ *implies* $|g(x) - g(u)| < \delta_f$*.]*
4. *If* f *is uniformly continuous on* $A \subseteq \mathbb{R}$*, and* $|f(x)| \geq k > 0$ *for all* $x \in A$*, show that* $1/f$ *is uniformly continuous on* A*.*

Bartle-144

Exercises 42.

1. *Prove that if f is uniformly continuous on a bounded subset A of $\mathbb{R}$, then f is bounded on A.*
2. *Show that the function $f(x) := 1/x^2$ is uniformly continuous on $A := [1, \infty)$, but that it is not uniformly continuous on $B := (0, \infty)$.*
3. *Show that if f is continuous on $[0, \infty)$ and uniformly continuous on $[a, \infty)$ for some positive constant a, then f is uniformly continuous on $[0, \infty)$. Using the result, one can show that $f(x) = \sqrt{x}$ is uniformly continuous on $[0, \infty)$.*
4. *Assume that g is defined on an open interval (a, c) and it is known to be uniformly continuous on $(a, b]$ and $[b, c)$, where $a < b < c$. Prove that g is uniformly continuous on (a, c).*

Exercises 43.

1. *Let $A \subseteq \mathbb{R}$ and suppose that $f : A \to \mathbb{R}$ has the following property: for each $\varepsilon > 0$ there exists a function $g_\varepsilon : A \to \mathbb{R}$ such that g_ε is uniformly continuous on A and $|f(x) - g_\varepsilon(x)| < \varepsilon$ for all $x \in A$. Prove that f is uniformly continuous on A.*
2. *A function $f : \mathbb{R} \to \mathbb{R}$ is said to be* **periodic** *on $\mathbb{R}$ if there exists a number $p > 0$ such that $f(x + p) = f(x)$ for all $x \in \mathbb{R}$. Prove that a continuous periodic function on $\mathbb{R}$ is bounded and uniformly continuous on $\mathbb{R}$.*
   [Hint : Since f is bounded on $[0, p]$, it follows that it is bounded on $\mathbb{R}$. Since f is continuous on $J := [-1, p + 1]$, it is uniformly continuous on J. Now show that this implies that f is uniformly continuous on $\mathbb{R}$.]

Bartle-144

Part - 2

Conservation by Uniformly Continuous Functions

Uniformly continuous functions have some very nice conserving properties. They map totally bounded sets onto totally bounded sets and Cauchy sequences onto Cauchy sequences.

On the other hand, uniformly continuous functions need not preserve boundedness.

Michael-150

Nice Properties of Uniformly Continuous Functions

We now discuss some nice properties of uniformly continuous functions.

Theorem 44.

Let $f : (X, d) \to (Y, \rho)$ be a map of metric spaces. If f is uniformly continuous on $A \subseteq X$ and (x_n) is a Cauchy sequence in A, then $(f(x_n))$ is a Cauchy sequence in Y.

Proof : Let (x_n) be a Cauchy sequence in A and let $\varepsilon > 0$. Since f is uniformly continuous on A, there exists $\delta > 0$ so that

$$x, y \in A \quad \text{and} \quad d(x, y) < \delta \quad \text{imply} \quad \rho(f(x), f(y)) < \varepsilon. \tag{1}$$

Since (x_n) is a Cauchy sequence, there exists $N_0 \in \mathbb{N}$ so that

$$d(x_n, x_m) < \delta \quad \text{for all } m, n > N_0.$$

From (1) we see that $m, n > N_0$ implies $\rho(f(x_n), f(x_m)) < \varepsilon$. This proves that $(f(x_n))$ is a Cauchy sequence in Y.

Nice Properties of Uniformly Continuous Functions

The preceding result gives us an alternative way of seeing that $f(x) := 1/x$ is not uniformly continuous on $(0,1)$.

We note that the sequence given by $x_n := 1/n$ in $(0,1)$ is a Cauchy sequence, but the image sequence, where $f(x_n) = n$, is not a Cauchy sequence.

We shall now use the relationship between the concepts of the limit of a function and that of a sequence. This relationship will be very useful in proving the existence of limit of a function in many cases.

Theorem 45.

Let $f : E \to \mathbb{R}$ where $E \subseteq \mathbb{R}$ be a function. Then $\lim_{x\to a} f(x) = \ell$ $(-\infty \le a \le \infty, -\infty \le \ell \le \infty)$ if and only if for every sequence $\{x_n\}$ in $S\backslash\{a\}$ with $x_n \to a$ as $n \to \infty$, we have $f(x_n) \to \ell$ as $n \to \infty$.

Nice Properties of Uniformly Continuous Functions

Theorem 46.

Let $f : E \subseteq \mathbb{R} \to \mathbb{R}$ be uniformly continuous on E. If x_0 is a limit point of E, then f has a limit at x_0.

Proof : Let x_0 be a limit point of S and let $\{x_n\}$ be any sequence in $S \backslash \{x_0\}$ converging to x_0. In view of Theorem 45, it is sufficient to prove that the sequence $\{f(x_n)\}$ is Cauchy (note that every Cauchy sequence of real numbers is convergent) and that all these sequences converge to the same limit. Since $\{x_n\}$ converges, it is Cauchy in E. As f is uniformly continuous on E, $\{f(x_n)\}$ is Cauchy.

Further, if $x_n \to x_0$ and $y_n \to x_0$, then the sequence $\{x_1, y_1, x_2, y_2 \ldots\}$ also converges to x_0 and hence the sequence $\{f(x_1), f(y_1), f(x_2), f(y_2) \ldots\}$ also converges. It now follows that $\{f(x_n)\}$ and $\{f(y_n)\}$ converge to the same limit.

Example 47.

Observe that Theorem 46 gives a necessary condition for uniform continuity but it is not a sufficient condition. For example, consider $g : \mathbb{R} \to \mathbb{R}$ *where* $g(x) = x^2$ *for all* $x \in \mathbb{R}$. *Clearly* g *is continuous on* $\mathbb{R}$ *and* g *has a limit at every limit point of* $\mathbb{R}$ *(note that* $\mathbb{R}$ *contains all its limit points) but* g *is not uniformly continuous.*

Karunakaran-5-29

Definition 48.

A function $f : X \to Y$ of metric spaces is said to be **Cauchy continuous** *if f carries every Cauchy sequence in X to a Cauchy sequence in Y.*

Exercise 49.

Show that every Cauchy continuous function $f : X \to Y$ is continuous. [Hint : For otherwise, we can find a point x_0, an $\varepsilon_0 > 0$, and a sequence (x_n) such that $d(x_n, x_0) \to 0$ but $\rho(f(x_n), f(x_0)) \geq \varepsilon_0$ for all n. Take $y_n = x_0$ if n is odd and take $y_n = x_n$ if n is even. Then (y_n) is a Cauchy sequence, but $(f(y_n))$ is not.]

A Cauchy continuous function may not be uniformly continuous. For example, $g(x) = x^2$ on $\mathbb{R}$.

A continuous function may not be Cauchy continuous. For example, $f(x) = 1/x$ on $(0, 1)$.

uniformly continuous $\subsetneq$ Cauchy continuous $\subsetneq$ continuous.

Totally Bounded

Definition 50.

Let (X, d) be a metric space and ε be an arbitrary positive number. Then a subset $A \subseteq X$ is said to be an **ε-net for** *X if given any $x \in X$, there exists a point $y \in A$ such that $d(x, y) < \varepsilon$.*

A **finite ε-net** *for X is an ε-net of X consisting of finite number of elements of X.*

Definition 51.

A subset $A \subseteq X$ is said to be **totally bounded** *if for every $\varepsilon > 0$, there exists a finite ε-net of A.*

- Every totally bounded set is bounded.
- A is totally bounded iff every sequence in A contains a Cauchy subsequence. (**Cauchy criterion for total boundedness**)
- Every compact metric space is totally bounded.
- A metric space is compact iff it is complete and totally bounded.

Theorem 52.

Suppose (X, d) and (Y, ρ) are metric spaces and $f : X \to Y$ is uniformly continuous on X. Then f maps every totally bounded subset of X to a totally bounded subset of Y.

Proof : Suppose S is a totally bounded subset of X. Suppose (y_n) is any sequence in $f(S)$. For each $n \in \mathbb{N}$, the subset $S \cap f^{-1}(\{y_n\})$ of X is non-empty.

We choose a sequence (x_n) with $x_n \in S \cap f^{-1}(\{y_n\})$ for each $n \in \mathbb{N}$. Then $f(x_n) = y_n$ for each $n \in \mathbb{N}$. By the Cauchy criterion for total boundedness of S, (x_n) has a Cauchy subsequence (x_{n_m}). Then, by what we have just proved, $(f(x_{n_m}))$, that is, (y_{n_m}), is a Cauchy subsequence of (y_n). Since (y_n) is an arbitrary sequence in $f(S)$, $f(S)$ satisfies the Cauchy criterion for total boundedness and so is totally bounded.

Not all continuous functions are sending totally bounded sets to totally bounded sets.

Example 53.

The function $x \mapsto 1/x$ *defined on* $(0, 1)$ *is continuous.*
But it maps the totally bounded subset $(0, 1)$ *of* $\mathbb{R}$ *to the closed unbounded subset* $(1, \infty)$ *of* $\mathbb{R}$*.*

Example 54.

The tangent function maps the totally bounded interval $(-\pi/2, \pi/2)$ *of* $\mathbb{R}$ *to the unbounded interval* $(-\infty, \infty)$*. It is not, as we already know, uniformly continuous on* $(-\pi/2, \pi/2)$*.*

Michael-151

A Nice Property of Uniformly Continuous

Theorem 55.

If f is a continuous function from a compact metric space (X, d) into a metric space (Y, ρ), then the range $f(X)$ of f is also compact.

As every uniformly continuous function on A is continuous on A, we have the following property.

Third property :

Suppose (X, d) and (Y, ρ) are metric spaces and $f : X \to Y$ is uniformly continuous on X. Then f maps every compact subset of X to a compact subset of Y.

Question

Do any or all of the above three properties characterize uniform continuity?

The answer is no.

For example, the exponential function satisfies all of them. It satisfies the first property because all Cauchy sequences are convergent in $\mathbb{R}$ and continuous functions map convergent sequences onto sequences that are convergent and therefore Cauchy. It satisfies the second property because, being continuous on $\mathbb{R}$, it is uniformly continuous on every bounded subset of $\mathbb{R}$. Such sets are totally bounded and so are mapped by the function to totally bounded subsets of $\mathbb{R}$. It satisfies the third property because it is uniformly continuous on closed bounded subsets of $\mathbb{R}$.

Michael-152

Question

Do uniformly continuous functions map all bounded sets onto bounded sets?

They do in familiar situations where the domain and codomain are subsets of $\mathbb{R}^n$, but that is because boundedness and total boundedness are the same thing in those spaces.

It is not always so.

Consider the identity function from $\mathbb{N}$ to $\mathbb{N}$, where the domain is given the discrete metric and the codomain the usual metric. The identity function is uniformly continuous because the metric on its domain is the discrete metric, but the domain is a bounded space and the range is not.

Michael-152

Uniform Continuity on Subsets of the Cantor Set

Subsets of the Cantor set are totally bounded since they are bounded subsets of $\mathbb{R}$. So **every uniformly continuous image of a subset of the Cantor set is also totally bounded**. But this is actually a characterization of totally bounded metric spaces.

Theorem 56.

Suppose (X, d) is a non-empty metric space. Then X is totally bounded if, and only if, there exists a bijective uniformly continuous function from a subset of the Cantor set K onto X.

Proof : Suppose X is totally bounded. For each $m \in \mathbb{N}$, choose a finite collection B_m of open balls of radius $1/m$ that covers X. All these balls together form a countable collection.

Michael-153

Proof (contd...)

By enumerating all the members of each B_m in turn, we form a sequence (U_n) of open balls in which, for each $m \in \mathbb{N}$, the balls of B_m precede those of B_{m+1}. [1] Then (U_n) has the property that $diam(U_n) \to 0$ as $n \to \infty$.

For each $x \in X$, let $\alpha_n(x) = 2$ if $x \in U_n$ and $\alpha_n(x) = 0$ otherwise, and set $g(x) = \sum_{n=1}^{\infty} \alpha_n(x)/3^n$. Then $g(x) \in K$. Note that there is an infinite number of values of n for which $\alpha_n(x) = 2$ because each B_m is a cover for X. It follows that g is injective because, for $x, z \in X$ with $x \neq z$, we have $\alpha_n(z) = 0$ whenever both $\alpha_n(x) = 2$ and $d(x, z) > diam(U_n)$.

Let $\phi = g^{-1}$. Then ϕ is a bijective map from the subset $g(X)$ of K onto X. We want to show that ϕ is uniformly continuous.

Michael-153

[1] Duplications are possible because radii are not well-defined, but this does not affect the argument.

Proof (contd...)

Let $\varepsilon > 0$ be given. Let $p \in \mathbb{N}$ be such that $p > 2/\varepsilon$. Then every member of B_p has diameter less than ε. Let $k \in \mathbb{N}$ be the largest subscript assigned to a member of B_p in the enumeration (U_n) of the covering balls.

Suppose a and b are arbitrary members of $g(X)$ that satisfy $|a - b| < 1/3^k$. Let $\phi(a) = x$ and $\phi(b) = z$. Then $g(x) = a$ and $g(z) = b$, so that $\alpha_n(x) = \alpha_n(z)$ for all $n \in \mathbb{N}_k$-in other words, for all $n \in \mathbb{N}_k$, $x \in U_n$ if, and only if, $z \in U_n$. Since B_p covers X and all members of B_p occur in the first k terms of (U_n), there exists $q \in \mathbb{N}_k$ with $z \in U_q \in B_p$, whence also $x \in U_q$.

Then $d(\phi(a), \phi(b)) = d(x, z) \leq diam(U_q) \leq 2/p < \varepsilon$. Since $\varepsilon > 0$ is arbitrary, the uniform continuity of ϕ follows. This proves the forward implication; the proof of the backward one is stated before the present result.

Michael-153

Uniform Continuity on Subsets of the Cantor Set

We deduce from Theorem 56 that all totally bounded metric spaces are relatively small. Since they are all in one-to-one correspondence with a subset of $\mathbb{R}$, none has cardinality greater than $\mathbb{R}$.

Michael-154

The next theorem involves extensions of functions. We say that a function g **is an extension of a function** f if $\mathrm{dom}(f) \subseteq \mathrm{dom}(g)$ and $f(x) = g(x)$ for all $x \in \mathrm{dom}(f)$.

Example 57.

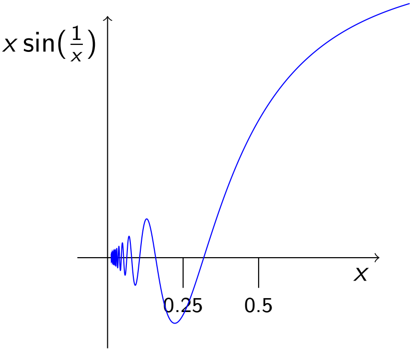
Let $f(x) = x\sin(\frac{1}{x})$ *for* $x \in (0, \frac{1}{\pi}]$. *The function defined by*

$$g(x) = \begin{cases} x\sin(\frac{1}{x}) & \text{for} \quad 0 < x \leq \frac{1}{\pi} \\ 0 & \text{for} \quad x = 0 \end{cases}$$

is an extension of f. *Note that* $\mathrm{dom}(f) = (0, \frac{1}{\pi}]$ *and* $\mathrm{dom}(g) = [0, \frac{1}{\pi}]$. *In this case,* g *is a continuous extension of* f.

Ross-139

Graph of $f(x) = x \sin(\frac{1}{x})$ for $x \in (0, \frac{1}{\pi}]$.

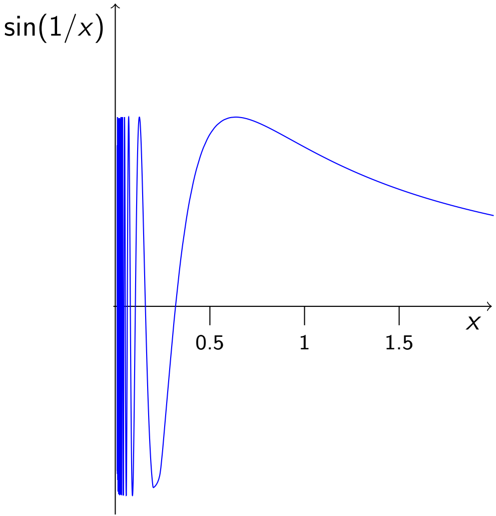


Example 58.

Let $f(x) = \sin(\frac{1}{x})$ *for* $x \in (0, \frac{1}{\pi}]$. *The function* f *can be extended to a function* g *with domain* $[0, \frac{1}{\pi}]$. *in many ways, but* g *will not be continuous.*

Ross-141

Graph of $f(x) = \sin(\frac{1}{x})$ for $x \in (0, \frac{1}{\pi}]$.



Theorem 59 (Uniform Extension Theorem).

Let X and Y be metric spaces and D be a dense subset of X. If $f : (D, d) \to (Y, \rho)$ is uniformly continuous and if Y is **complete**, *then there exists a uniformly continuous function $g : X \to Y$ such that $g(x) = f(x)$ for all $x \in D$.*

Steps involved:

1. For $x \in X$, take a sequence (x_n) in D such that $x_n \to x$.
2. Observe that $f(x_n)$ is a Cauchy sequence in Y.
3. Write $g(x) = \lim_{n\to\infty} f(x_n)$.
4. Observe that if (x'_n) in D with $x'_n \to x$, then $\lim_{n\to\infty} f(x'_n) = \lim_{n\to\infty} f(x_n)$.
5. Observe that $g : X \to Y$ is well-defined.
6. Observe that $g(x) = f(x)$ for every $x \in D$.
7. Show that g is uniformly continuous on X.

Proof of the theorem

Let $x \in X$. Then by density of D in X, there exists a sequence (x_n) in D such that $x_n \to x$. Note that (x_n) is Cauchy (since every convergent sequence is Cauchy). Since f is uniformly continous, the sequence $(f(x_n))$ is Cauchy in Y. Since Y is complete, there exists $y \in Y$ such that $f(x_n) \to y$.

We set $g(x) = y$. **We need to show that $g(x)$ is well-defined in the sense that if (x_n') in D converges to x and if $f(x_n') \to y'$, then $y = y'$.**

Suppose there is a sequence (x_n') in D converging to x and $f(x_n')$ converges to some y' in X. We need to show that $y = y'$.

Since f is uniformly continuous on D and $d(x_n, x_n') \to 0$ as $n \to \infty$, $\rho(f(x_n), f(x_n')) \to 0$ as $n \to \infty$.

Proof (contd...)

Let $\varepsilon > 0$ be given. Since each of the sequences $\rho(f(x_n), y)$, $\rho(f(x'_n), y')$ and $\rho(f(x_n), f(x'_n))$ is converging 0 as $n \to \infty$, there are natural numbers N_1, N_2, N_3 such that

$$\begin{aligned} \rho(f(x_n), y) &< \frac{\varepsilon}{3}, \quad \text{for all } n \geq N_1 && (1) \\ \rho(f(x'_n), y') &< \frac{\varepsilon}{3}, \quad \text{for all } n \geq N_2 && (2) \\ \rho(f(x_n), f(x'_n)) &< \frac{\varepsilon}{3}, \quad \text{for all } n \geq N_3. && (3) \end{aligned}$$

Let $N_0 = \max\{N_1, N_2, N_3\}$.

Then by the triangle inequality we have for all $n \geq N_0$

$$\rho(y, y') \leq \rho(y, f(x_n)) + \rho(f(x_n), f(x'_n)) + \rho(f(x'_n), y') < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

Since LHS of the above inequality is independent of n, we have $\rho(y, y') < \varepsilon$, for every $\varepsilon > 0$. Thus $y = y'$.

Proof (contd...)

We proved that the map g is well-defined. Note that $g(x) = f(x)$ for $x \in D$, as we may take the constant sequence $(x_n := x)$ convergent to x.

Next prove that g is uniformly continuous on X.

Let $\varepsilon > 0$ be given. Since f is uniformly continuous on D, there exists $\delta > 0$ such that for all $a, b \in D$ with $d(a, b) < \delta$, we have $\rho(f(a), f(b)) < \varepsilon$. We have to show the samething happens for any two points in X. Let $x, y \in X$ with $d(x, y) < \delta$.

Since D is dense in X, there are sequences (x_n) and (y_n) in D such that $x_n \to x$ and $y_n \to y$. In particular, (x_n) and (y_n) are Cauchy. Since f is uniformly continuous on D, $(f(x_n))$ and $(f(y_n))$ are Cauchy sequences in Y, converging to some u and v respectively.

Proof (contd...)

Hence by the above arguments, we have $g(x) = u, g(y) = v$.

We need to show that $\rho(g(x), g(y)) = \rho(u, v) < \varepsilon$. We use $\varepsilon/3$-argument. Now,

$$\begin{aligned} \rho(u, v) &\leq \rho(u, f(x_n)) + \rho(f(x_n)), f(y_n)) + \rho(f(y_n), v) \\ &< \varepsilon \quad \text{for all } n \geq \max\{N_1, N_2\}. \end{aligned}$$

Please note that we got natural numbers N_1 and N_2 from the convergence of sequences $\rho(u, f(x_n))$ and $\rho(f(y_n), v)$ to 0 respectively (we apply $\varepsilon/3$-argument) ; also $\rho(f(x_n)), f(y_n)) < \varepsilon/3$ (applying uniform continuity of f on D).

LHS is independent of n. Hence

$$\rho(g(x), g(y)) < \varepsilon.$$

This completes the proof.

Application of the uniform extension theorem

An instructive application of the theorem is the extension of the meaning of a^r for $r \in \mathbb{Q}$ to a^x for any $x \in \mathbb{R}$ for any fixed $a > 0$.

Starting from the existence of nth roots (Theorem A), one assigns a meaning to $a^{m/n}$ where $m, n \in \mathbb{Z}$ and $n \neq 0$. One also verifies the laws of exponents hold. We shall assume these results in the proof of the theorem below.

Theorem A : [Existence of nth roots] Let $\alpha \in [0, \infty)$ and $n \in \mathbb{N}$. Then there exists a unique $x \in [0, \infty)$ such that $x^n = \alpha$.

Theorem 60.

Fix a positive $a \in \mathbb{R}$ and $N \in \mathbb{N}$. Then the function $r \longmapsto a^r$ from $\mathbb{Q} \cap [-N, N]$ to $\mathbb{R}$ is uniformly continuous. Hence it extends to a continuous function from $[-N, N]$ to $\mathbb{R}$. This function is denoted by a^x for $x \in [-N, N]$.

Proof : Let $x, x+h \in \mathbb{Q} \cap [-N, N]$. We estimate

$$|a^{x+h} - a^x| = a^x |a^h - 1| \leq a^N |a^h - 1|.$$

If we show that $a^h \to 1$ as $h \to 0$ in $\mathbb{Q}$, then we are through. (Why? How does the uniform continuity follow?) This follows from the fact that $a^{1/n} \to 1$ as $n \to \infty$ (How?) We also used the fact that $x \longmapsto a^x$ is an increasing function on $\mathbb{Q}$ (Where ?) Prove this.

Remark 61.

Since a^x defined on various $[-N, N]$ coincide on their common domain, it follows that we have a function $x \longmapsto a^x$ for all $x \in \mathbb{R}$.

Kumaresan-78

Continuous Extension Theorem

We have seen examples of functions that are continuous but not uniformly continuous on open intervals; for example, the function $f(x) = 1/x$ on the interval $(0, 1)$.

On the other hand, by the uniform continuity theorem, a function that is continuous on a closed bounded interval is always uniformly continuous. So the question arises: Under what conditions is a function **uniformly continuous on a bounded open interval ?**

The answer reveals the strength of uniform continuity, for it will be shown that a function on (a, b) is uniformly continuous iff it can be defined at the end points to produce a function that is continuous on the closed interval.

Bartle-140

Theorem 62 (Continuous Extension Theorem).

A function $f : (a, b) \to \mathbb{R}$ is uniformly continuous on the interval (a, b) if and only if it can be defined at the endpoints a and b such that the extended function is continuous on $[a, b]$.

Proof : $(\Leftarrow)$ This direction is trivial.
$(\Rightarrow)$ Suppose f is uniformly continuous on (a, b). We shall show how to extend f to a; the argument for b is similar. This is done by showing that $\lim\limits_{x \to c} f(x) = L$ exists, and this is accomplished by using the sequential criterion for limits. If (x_n) is a sequence in (a, b) with $\lim(x_n) = a$, then it is a Cauchy sequence, and by the preceding theorem, the sequence $(f(x_n))$ is also a Cauchy sequence, and so is convergent. Thus the limit $\lim(f(x_n)) = L$ exists.

Bartle-140

Proof (contd...)

If (u_n) is any other sequence in (a, b) that converges to a, then $\lim(u_n - x_n) = a - a = 0$, so by the uniform continuity of f we have

$$\begin{aligned} \lim(f(u_n)) &= \lim(f(u_n) - f(x_n)) + \lim(f(x_n)) \\ &= 0 + L = L. \end{aligned}$$

Since we get the same value L for every sequence converging to a, we infer from the sequential criterion for limits that f has limit L at a. If we define $f(a) := L$, then f is continuous at a.

The same argument applies to b, so we conclude that f has a continuous extension to the interval $[a, b]$.

Bartle-140

Uniform Extension Theorem

Example 63.

Since the limit of $f(x) := \sin(1/x)$ *at 0 does not exist, we infer from the Continuous Extension Theorem that the function is not uniformly continuous on* $(0, b]$ *for any* $b > 0$.

On the other hand, since $\lim_{x\to 0} x\sin(1/x) = 0$ *exists, the function* $g(x) := x\sin(1/x)$ *is uniformly continuous on* $(0, b]$ *for all* $b > 0$.

Bartle-140

Equicontinuity

We have discussed that in the case of uniform continuity of f, given $\varepsilon > 0$, the choice of $\delta > 0$ depends only on ε, not on the point.

In the following definition, we have a family $\mathcal{F}$ of real functions defined on a common set $E \subseteq \mathbb{R}$ in which the choice of $\delta > 0$ does not depend on either the point or the funciton $f \in \mathcal{F}$. **In particular, every function f in the family $\mathcal{F}$ is uniformly continuous.**

Definition 64.

A family $\mathcal{F}$ of real functions defined on a common set $E \subseteq \mathbb{R}$ is said to be **equicontinuous on** *E if given $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ for all $x, y \in E$ with $|x - y| < \delta$ and for all $f \in \mathcal{F}$.*

We shall later discuss equicontinuity.

Karunakaran-9-22

Exercise 65.

Show that the uniform limit of a sequence of uniformly continuous functions on $\mathbb{R}$ is uniformly continuous on $\mathbb{R}$.

Karunakaran-9-39

Uniform Continuity for Functions of Two Variables

The notion of uniform continuity for functions of one variable can be easily extended to functions of two variables. Let D be a subset of $\mathbb{R}^2$. A function $f : D \to \mathbb{R}$ is said to be **uniformly continuous** on D if for any sequences $((x_n, y_n))$ and $((u_n, v_n))$ in D such that $|(x_n, y_n) - (u_n, v_n)| \to 0$, we have $|f(x_n, y_n) - f(u_n, v_n)| \to 0$. Note that $|.|$ denotes Euclidean distance in $\mathbb{R}^2$.

Specializing one of the two sequences to a constant sequence, we readily see that a uniformly continuous function is continuous. As in the case of functions of one variable, the converse is true if the domain is closed and bounded.

Proposition 66.

Let $D \subseteq \mathbb{R}^2$ be a closed and bounded set. Then every continuous function on D is uniformly continuous on D.

Proof : Suppose $f : D \to \mathbb{R}$ is continuous but not uniformly continuous on D. Then there are sequences $((x_n, y_n))$ and $((u_n, v_n))$ in D such that $|(x_n, y_n) - (u_n, v_n)| \to 0$, but $|f(x_n, y_n) - f(u_n, v_n)| \nrightarrow 0$, The latter implies that there are $\varepsilon > 0$ and positive integers $n_1 < n_2 < \cdots$ such that $|f(x_{n_k}, y_{n_k}) - f(u_{n_k} - v_{n_k}| \geq \varepsilon$ for all $k \in \mathbb{N}$. Now, by the Bolzano-Weierstarass Theorem, $((x_{n_k} - y_{n_k}))$ has a convergent sub sequence, say $((x_{n_{k_j}}, y_{n_{k_j}}))$. If $(x_{n_{k_j}}, y_{n_{k_j}}) \to (x_0, y_0)$, then $(x_{n_{k_j}}, y_{n_{k_j}}) \to (x_0, y_0)$, because $|(x_n, y_n) - (u_n, v_n)| \to 0$. Since f is continuous on D, we see that $|f(x_{n_{k_j}}, y_{n_{k_j}}) - f(u_{n_{k_j}}, v_{n_{k_j}})| \to |f(x_0, y_0) - f(x_0, y_0)| = 0$. But this is a contradiction, since $|f(x_{n_{k_j}}, y_{n_{k_j}}) - f(u_{n_{k_j}}, v_{n_{k_j}})| \geq \varepsilon$ for all $j \in \mathbb{N}$. Sudhir 61-63

Uniform Continuity for Functions of Two Variables

Example 67.

(i) *Consider* $f : \mathbb{R}^2 \to \mathbb{R}$ *defined by* $f(x,y) := x + y$. *Then it is clear that* f *is uniformly continuous on* $\mathbb{R}^2$.

(ii) *If* $D \subseteq \mathbb{R}^2$ *and* $f : D \to \mathbb{R}$ *is uniformly continuous, then for every fixed* $(x_0, y_0) \in D$, *the functions* $\phi : D_1 \to \mathbb{R}$ *and* $\psi : D_2 \to \mathbb{R}$ *(of one variable) defined by*

$$\phi(x) := f(x, y_0) \text{ for } x \in D_1 \quad \text{and } \psi(y) := f(x_0, y) \text{ for } y \in D_2$$

are uniformly continuous. This follows from the definition of uniform continuity by specializing one of the coordinates in the two sequences to a constant sequence.

Sudhir 61-63

Example 68.

Consider $D \subseteq \mathbb{R}^2$ *and* $f : D \to \mathbb{R}$ *given by* $D := \{(x, y) \in \mathbb{R}^2 : x, y \in [0, 1]$ *and* $(x, y) \neq (0, 0)\}$ *and* $f(x, y) := \frac{1}{x+y}$. *Then* f *is continuous on* D *but not uniformly continuous on* D. *To see the latter, consider the sequences* $((x_n, y_n))$ *and* $((u_n, v_n))$ *in* D *given by* $(x_n, y_n) := (1/n, 0)$ *and* $(u_n, v_n) := (1/(n+1), 0)$ *for* $n \in \mathbb{N}$. *We have* $|(x_n, y_n) - (u_n, v_n)| = 1/n(n+1) \to 0$, *but* $|f(x_n, y_n) - f(u_n, v_n)| = |n - (n+1)| = 1 \nrightarrow 0$. *Alternatively, we could use (ii) above and the fact that* $\phi : (0, 1] \to \mathbb{R}$ *defined by* $\phi(x) = f(x, 0) = 1/x$ *is not uniformly continuous on* $(0, 1]$. *It may be noted here that the domain of* f *is bounded but not closed.*

Sudhir 61-63

Example 69.

Consider $f : \mathbb{R}^2 \to \mathbb{R}$ *defined by* $f(x, y) : x^2 + y^2$. *Then* f *is continuous on* $\mathbb{R}^2$, *but not uniformly continuous on* $\mathbb{R}^2$. *To see the latter, consider the sequences* $((x_n, y_n))$ *and* $((u_n, v_n))$ *in* D *given by* $(x_n, y_n) := (n, 0)$ *and* $(u_n, v_n) := (n - (1/n), 0)$ *for* $n \in \mathbb{N}$. *We have*
$|x_n, y_n) - (u_n, v_n)| = 1/n \to 0$, *but*
$|f(x_n, y_n) - f(u_n, v_n)| = |n^2 - [n^2 - 2 + (1/n^2)]]) = 2 - (1/n^2) \nrightarrow 0.$
Alternatively, we could use (ii) above and the fact that $\phi : \mathbb{R} \to \mathbb{R}$ *defined by* $\phi(x) = f(x, 0) = x^2$ *is not uniformly continuous on* $\mathbb{R}$. *It may be noted here that the domain of* f *is closed but not bounded. On the other hand, the restriction of* f *to any bounded subset of* $\mathbb{R}^2$ *is uniformly continuous.*

Sudhir 61-63

A criterion for the uniform continuity of a function of two variables that does not involve convergence of sequences can be given as follows.

Proposition 70.

Let $D \subseteq \mathbb{R}^2$. Consider a function $f : D \to \mathbb{R}$. Then f is uniformly continuous on D if and only if it satisfies the following $\varepsilon - \delta$ condition: For every $\varepsilon > 0$, there is $\delta > 0$ such that
$(x, y, (u, v) \in D$ and$|(x, y) - (u, v)| < \delta \Rightarrow |f(x, y) - f(u, v)| < \varepsilon$.

Proof : Assume that f is uniformly continuous on D. Suppose the $\varepsilon - \delta$ condition does not hold. Then there id $\varepsilon > 0$ such that for any $\delta > 0$, we can find $(x, y), (u, v) \in D$ for which $|(x, y) - (u, v)| < \delta$, but $|f(x, y) - f(u, v)| \geq \varepsilon$.

Sudhir 61-63

Proof (contd...)

Considering $\delta := 1/n$ for $n \in \mathbb{N}$, we obtain sequences $((x_n, y_n))$ and $((u_n, v_n))$ in D such that $|(x_n, y_n) - (u_n, v_n)| < \frac{1}{n}$ and $|f(x_n, y_n) - f(u_n, v_n)| \geq \varepsilon$ for all $n \in \mathbb{N}$. Consequently, $|(x_n, y_n) - (u_n, v_n)| \to 0$, but $|f(x_n, y_n) - f(u_n, v_n)| \nrightarrow 0$. this contradicts the assumption that f is uniformly continuous on D.

Conversely, assume that the $\varepsilon - \delta$ condition is satisfied. Suppose $((x_n, y_n))$ and $((u_n, v_n))$ are any sequences in D such that $|(x_n, y_n) - (u_n, v_n)| \to 0$. Let $\varepsilon > 0$ be given. Then there is $\delta > 0$ such that if $(x, y), (u, v) \in D$ satisfy $|(x, y) - (u, v)| < \delta$, then $|f(x, y) - f(u, v) < \varepsilon$. Now, for this $\delta > 0$, we can find $n_0 \in \mathbb{N}$ such that $|(x_n, y_n) - (u_n, v_n)| < \delta$ for all $n \geq n_0$. Consequently, $|f(x_n, y_n) - f(u_n, v_n)| < \varepsilon$ for all $n \geq n_0$. Thus $|f(x_n, y_n) - f(u_n, v_n)| \to 0$. This proves the uniform continuity of f on D.

Sudhir 61-63

The type of global continuity that we habitually encounter amongst linear maps between normed linear spaces (in Functional Analysis) is Lipschitz continuity. It is stronger than uniform continuity and has the advantage that it preserves boundedness.

Definition 71.

Let X and Y be metric spaces. A function $f : X \to Y$ is said to be **Lipschitz function** *if there exists $k > 0$ such that*

$$\rho(f(x), f(y)) \leq k\ d(x, y) \quad \forall x, y \in X. \qquad (1)$$

The number k above is called a **Lipschitz constant**. *Every real number larger than k is also a Lipschitz constant for f.*
If $f : X \to Y$ is a Lipschitz function function with Lipschitz constant $k < 1$, then f is called a **contraction**.

Isometry

Definition 72.

Let (X, d) and (Y, ρ) be metric spaces and $f : X \to Y$. The function f is said to be **isometry** *if $d(x, y) = \rho(f(x), f(y))$ for all $x, y, \in X$.*

1. Isometry is necessarily injective and its inverse is also an isometry.
2. As metric spaces, X and $f(X)$ are **indistinguishable**; $f(X)$ is merely a relabelling of X.
3. Every metric space has a completion and any two completions are isometric to each other.

Example 73.

Isometries behave much better than other Lipschitz functions, as they obey an equality rather than an inequality.

Lipschitz Function

Let f be a real-valued function whose domain $\mathbb{I}$ is an interval.

The condition (1) that a function $f : \mathbb{I} \to \mathbb{R}$ on an interval $\mathbb{I}$ is a Lipschitz function can be interpreted geometrically as follows. If we write the condition as

$$\left| \frac{f(x) - f(u)}{x - u} \right| \leq k \quad \text{for all } x, y \in \mathbb{I}, x \neq u,$$

then the quantity inside the absolute values is the slope of a line segment joining the points $(x, f(x))$ and $(u, f(u))$. Thus a function f satisfies a Lipschitz condition if and only if the slopes of all line segments joining two points on the graph of $y = f(x)$ over $\mathbb{I}$ are bounded by some number k.

Bartle-139, Stephen-120

Every Lipschitz function function is uniformly continuous.

Theorem 74.

Suppose (X, d) and (Y, ρ) are metric spaces, A is a subset of X and $f : X \to Y$.

(i) *If f is a Lipschitz function on A with Lipschitz constant $k > 0$, then f is uniformly continuous on A and δ in the definition of uniform continuity can be taken to be ε/k.*

(ii) *If f is a Lipschitz function on X with Lipschitz constant $k > 0$, then f is a Lipschitz function on A with Lipschitz constant k.*

Proof : For every $x, y \in X$, we have

$$d(x, y) < \varepsilon/k \Rightarrow \rho(f(x), f(y)) \leq k\, d(x, y) < \varepsilon,$$

which proves (i). The proof of (ii) is obvious.

Michael-154

Example 75.

If $f(x) := x^2$ *on* $A := [0, b]$, *where* $b > 0$, *then*

$$|f(x) - f(y)| = |x + y||x - y| \leq 2b\,|x - y|$$

for all x, y *in* $[0, b]$. *Thus* f *satisfies the Lipschitz condition with* $k = 2b$ *on* A, *and therefore* f *is uniformly continuous on* A.

Note that f *does not satisfy the Lipschitz condition (1) on the interval* $[0, \infty)$.

Exercise 76 (Compositions of Lipschitz functions are Lipschitz).

Suppose $(X, d), (Y, \rho)$ and (Z, γ) are metric spaces, $f : X \to Y$ and $g : Y \to Z$. If f and g are Lipschitz functions with Lipschitz constants k and ℓ on X and $f(X)$, respectively, then $g \circ f$ is a Lipschitz function on X with Lipschitz constant $k\ell$.

Michael-158

Not every uniformly continuous function is a Lipschitz function.

Example 77.

Let $g(x) := \sqrt{x}$ *for* x *in the closed bounded interval* $I := [0, 2]$*. Since* g *is continuous on* I*, it follows from the Uniform Continuity Theorem that* g *is uniformly continuous on* I*.*

If $|g(x) - g(0)| \leq k\ |x - 0|$ *for all* $x \in [0, 1]$*, then* $\sqrt{x} \leq k\ x$ *for* $x \in [0, 1]$*. But if* $x_n := 1/n^2$*, then* k *must satisfy* $n \leq k$ *for all* $n \in \mathbb{N}$*, which is impossible.*

Hence, there is no number $k > 0$ *such that* $|g(x)| \leq k\ |x|$ *for all* $x \in I$*.*

Therefore, g *is not a Lipschitz function on* I*.*

Bartle-139

Lipschitz Function

The Uniform Continuity Theorem and Theorem 74 (every Lipschitz function is uniformly continuous) can sometimes be combined to establish the uniform continuity of a function on a set.

Example 78.

We consider $g(x) := \sqrt{x}$ *on the set* $A := [0, \infty)$. *The uniform continuity of* g *on the interval* $I := [0, 2]$ *follows from the Uniform Continuity Theorem. If* $J := [1, \infty)$, *then if both* x, y *are in* J, *we have*

$$|g(x) - g(y)| = |\sqrt{x} - \sqrt{y}| = \frac{|x - y|}{\sqrt{x} + \sqrt{y}} \leq \frac{1}{2}|x - y|.$$

Thus g *is a Lipschitz function on* J *with constant* $k = \frac{1}{2}$, *and hence by Theorem 74,* g *is uniformly continuous on* $[1, \infty)$. *Since* $A = I \cup J$, *it follows that* g *is uniformly continuous on* A *by taking* $\delta(\varepsilon) := \inf\{1, \delta_I(\varepsilon), \delta_J(\varepsilon)\}$.

Bartle-139

Theorem 79.

Suppose (X, d) and (Y, ρ) are metric spaces, S is a bounded subset of X and $f : X \to Y$ is a Lipschitz function. Then $f(S)$ is bounded in Y.

Proof : Let $k > 0$ be a Lipschitz constant for f and suppose $x, y \in S$.

Then $\rho(f(x), f(y)) \leq k\ d(x, y) \leq k\ diam(S)$, so that

$$diam(f(S)) \leq k\ diam(S).$$

Michael-154

Continuity on Unions

A function f is continuous if, and only if, it is continuous at every point of its domain; the same applies to restrictions of f. Let us suppose that the domain of f is split up into several constituent parts and that the restriction of f to each of those parts is continuous; in other words, each restriction of f is continuous at every point of the appropriate constituent part.

Does it follow that f is continuous at every point of its domain and is therefore a continuous function? It is important to know that it does not, even if the constituent parts are closed in the domain and mutually disjoint (8.3.7). However, 8.7.1 gives a sufficient condition for the truth of the implication.

Continuity on Unions

Theorem 80.

Suppose (X, d) and (Y, ρ) are metric spaces. Suppose $\mathcal{C}$ is non-empty collection of mutually disjoint non-empty subspaces of X and $f : \bigcup \mathcal{C} \to Y$. Suppose that, for each $A \in \mathcal{C}$, we have $A \cap Cl(\bigcup(\mathcal{C}\ \{A\})) = \phi$ and $f|_A$ continuous. Then f is continuous on $\bigcup \mathcal{C}$.

Proof : Suppose $a \in \bigcup \mathcal{C}$, and let $A \in \mathcal{C}$ be such that $a \in A$. If $\mathcal{C} = \{A\}$, the result is trival, so we suppose otherwise. Let $\varepsilon > 0$. Since $f|_A$ is continuous, it is continuous at a, so there exists $\gamma > 0$ such that, for all $x \in A$ for which $d(x, a) < \gamma$, we have $\rho(f(x), f(a)) < \varepsilon$.

Michael-138

Continuity on Unions

Since $A \cap Cl(\bigcup(\mathcal{C} \setminus \{A\})) = \phi$, 3.6.10 given $\eta = \text{dist}(a, \bigcup(\mathcal{C} \setminus \{A\})) \neq 0$, and, because A is not the only member of $\mathcal{C}$, $\eta > 0$. Let $\delta = \min\{\gamma, \eta\}$. Then, for each $x \in \bigcup \mathcal{C}$ with $d(x, a) < \delta$, we have $x \in A$, so that $e(f(x), f(a)) < \varepsilon$.

Because $\varepsilon > 0$ is arbitrary, f is continuous at a; and because a is arbitrary in $\bigcup \mathcal{C}$, f satisfies the local criterion for continuity on $\bigcup \mathcal{C}$, so f is continuous on $\bigcup \mathcal{C}$. Michael-138

Question 8.7.2

In 8.3.7 and 8.7.1, we considered only disjoint subsets of a domain. Let us look now at overlapping parts of a domain. Suppose X and Y are metric spaces and A and B are subsets X with $A \cap B \neq \phi$. Suppose $f : A \cup B \to Y$ has continuous restrictions to A and B. Does f have to be continuous? The answer is no. Consider the real function defined on $\mathbb{C}$ by

$$z \mapsto \begin{cases} 0, & \text{if } \Re \geq 0 \text{ and } \Im z \geq 0; \\ |\Re(z)|, & \textit{if } \Re z < 0 \text{ and } \Im z \geq 0; \\ |\Im(z)|, & \textit{if } \Im z < 0. \end{cases}$$

This function is continuous on $\{z \in \mathbb{C} | \Im z \geq 0\}$ and is also continuous on $\{z \in \mathbb{C} | \Re z \geq 0 \text{ or } \Im z < 0\}$, but, being discontinuous at every point of the negative part of the real line, is not continuous on their union $\mathbb{C}$.

Michael-138

Every finite product of metric spaces comes equipped with natural projections onto the coordinate spaces (1.6). These projections are continuous provided only that the product is endowed with a product metric. Moreover, a function that maps into the product is continuous if, and only if, its compositions with the natural projections are all continuous.

Theorem 81.

Suppose $n \in \mathbb{N}$ and, for each $i \in \mathbb{N}_n, (X_i, \tau_i)$ is a space. Endow $P = \Pi_{i=1}^n X_i$ with a product metric. Then, for each $j \in \mathbb{N}_n$, the natural projection $\pi_j : P \to Z_j$ is continuous.

Proof : Suppose $j \in \mathbb{N}_n$ and V is open in X_j. Then $\pi_j^{-1}(V) = \{x \in P | x_j \in V\}$, which can be expressed as $\Pi_{i=1}^n U_i$, where $U_j = V$ and $U_i = X_i$ for all $i \in \mathbb{N}_n \ \{j\}$. This is certainly a member of the product topology (4.5.2) because V is open in X_j and X_i is open in X_i for all $i \in \mathbb{N}_n \backslash \{j\}$. Therefore π_j is continuous.

Theorem 82.

Suppose $n \in \mathbb{N}$ and, for each $i \in \mathbb{N}_n, (X_i, \tau_i)$ is a metric space. Endow $P = \Pi_{i=1}^{n} X_i$ with a product metric. Suppose Z is a metric space and $f : Z \to P$. Then f is continuous if, and only if, $\pi \circ f$ is continuous for all $i \in \mathbb{N}_n$.

A function that is uniformly continuous on a number of disjoint closed sets may well not be uniformly continuous on their union even if the condition of Theorem 8.7.1 is satisfied; for a sufficient condition for uniform continuity, we confine ourselves to finite unions. Lipschitz continuity has even less stability.

Theorem 83.

Suppose (X, d) and (Y, ρ) are metric spaces and $\mathcal{C}$ is a finite collection of non-empty subsets of X such that dist$(A, B) > 0$ for all $A, B \in \mathcal{C}$. Suppose $f : \bigcup \mathcal{C} \to Y$ has uniformly continuous restriction to each member of $\mathcal{C}$. Then f is uniformly continuous on $\bigcup \mathcal{C}$.

Proof of the theorem

Let $\varepsilon > 0$ be given. For each $A \in \mathcal{C}$, $f|_A$ is uniformly continuous by hypothesis. Let δ_A be such that, for all

$$u, v \in A, d(u, v) < \delta_A \quad \Rightarrow \quad \rho(f(u), f(v)) < \varepsilon.$$

The set

$$\{dist(A, B)|A, B \in \mathcal{C}\} \cup \{\delta_A|A \in \mathcal{C}\}$$

is a finite subset of the set of all positive real numbers and so has a minimum member $\delta > 0$.

Then, for all $u, v \in \bigcup \mathcal{C}$ with $d(u, v) < \delta$, there exists $A \in \mathcal{C}$ such that $u, v \in A$ and then, since $\delta \leq \gamma_A$, we have $\rho(f(u), f(v)) < \varepsilon$.

Michael-158

Example 84.

The condition dist$(A, B) > 0$ in Theorem 83 cannot in general be weakened to $\overline{A} \cap \overline{B} = \emptyset$. Consider the two subsets $A = \{(x, 1/x) | x \in \mathbb{R} \setminus \{0\}\}$ and $B = \{(x, 0) | x \in \mathbb{R}\}$ of $\mathbb{R}^2$ with the Euclidean metric.
Both are closed in $\mathbb{R}^2$. Define f to be 1 on A and 0 on B. Then f is uniformly continuous on each of A and B but not on $A \cup B$. Specifically, let $\delta > 0$. Then, for $x > 1/\delta$, the distance from $(x, 1/x)$ to $(x, 0)$ is less than δ and the distance between their images under f is 1.

Michael-158

Example 85.

No theorem like Theorem 83 is possible for Lipschitz functions. Consider the function $f : \mathbb{N} \to \mathbb{N}$ *given by* $2n - 1 \mapsto 2n - 1$ *and* $2n \mapsto 4n$ *for each* $n \in \mathbb{N}$, *where* $\mathbb{N}$ *is endowed, as domain and codomain, with its usual metric.*

The restriction of f *to the odd natural numbers is the identity function, which is Lipschitz with Lipschitz constant* 1; *and the restriction of* f *to the even natural numbers is the doubling function, which is Lipschitz with Lipschitz constant* 2. *These two sets are a distance* 1 *apart.*

But f *is not Lipschitz because for each* $k > 0$ *and* $n \in \mathbb{N}$ *with* $n \geq k/2$, *we have* $f(2n) - f(2n-1) = 2n + 1 > k(2n - (2n-1))$.

Michael-158

Exercise 86.

Find a function that is uniformly continuous on an infinite number of closed sets, every two of which are of distance at least 1 *from each other, but is not uniformly continuous on their union.*

Michael-162

Continuity of Mappings into Product Spaces

Every finite product of metric spaces comes equipped with natural projections onto the coordinate spaces. These projections are continuous provided only that the product is endowed with a product metric. Moreover, a function that maps into the product is continuous iff its compositions with the natural projections are also continuous.

Theorem 87.

Suppose $n \in \mathbb{N}$ and for each $i \in \mathbb{N}_n$, (X_i, τ_i) is a metric space. Endow $P = \prod_{i=1}^{n} X_i$ with a product metric. Then, for each $j \in \mathbb{N}$, the natural projection $\pi_j : P \to X_j$ is continuous.

Theorem 88.

Suppose $n \in \mathbb{N}$ and for each $i \in \mathbb{N}_n$, (X_i, τ_i) is a metric space. Endow $P = \prod_{i=1}^{n} X_i$ with a product metric. Suppose Z is a metric space and $f : Z \to P$. Then f is continuous iff $\pi_i \circ f$ is continuous, for all $i \in \mathbb{N}_n$.

Not every product metric ensures the uniform continuity of the natural projections. We need to make restrictions in order to get theorems similar to Theorem 87 and Theorem 88.

Theorem 89.

Suppose $n \in \mathbb{N}$ and, for each $i \in \mathbb{N}_n, (X_i, \tau_i)$ is a metric space. Endow $P = \Pi_{i=1}^n X_i$ with a conserving metric e. Then all the natural projections $\pi_i : P \to X_i$ are Lipschitz maps with Lipschitz constant 1.

Proof : Suppose $a, b \in P$. Then, for each $i \in \mathbb{N}_n, \tau_i(\pi_i(a), \pi_i(b)) \leq \rho(a, b)$ because e is a conserving metric.

Michael-159

Theorem 90.

Suppose $n \in \mathbb{N}$ *and, for each* $i \in \mathbb{N}_n, (X_i, \tau_i)$ *is a metric space. Endow* $P = \prod_{i=1}^n X_i$ *with a conserving metric* e*. Suppose* (Z, m) *is a metric space and* $f : Z \to P$*. Then:*

(i) f *is uniformly continuous if, and only if,* $\pi_i \circ f$ *is uniformly continuous for all* $i \in \mathbb{N}_n$*.*

(ii) f *is a Lipschitz function if, and only if, and only if,* $\pi_i \circ f$ *is a Lipschitz function for all* $i \in \mathbb{N}_n$*.*

Michael-159

Proof of the theorem

The forward implications are immediate consequences of Theorem 89, Exercise 38 and Exercise 76. For the backward implication in (i), suppose $\pi_i \circ f$ is uniformly continuous for each $i \in \mathbb{N}_n$.

Let $\varepsilon > 0$ and, for each $i \in \mathbb{N}_n$, let γ_i be such that, for each $a, b \in Z$, we have $m(a, b) < \gamma + i \Rightarrow \tau_i(\pi_i(f(a)), \pi_i(f(b))) < \varepsilon/n$. Let $\delta = \min\{\gamma_i | i \in \mathbb{N}_n\}$. Then $m(a, b) < \delta \Rightarrow \sum_{i=1}^{n} \tau_i(\pi_i(f(a)), \pi_i(f(b))) < \varepsilon$ and, because ρ is a conserving metric, we have also $m(a, b) < \delta \Rightarrow \rho(f(a), f(b)) < \varepsilon$, as required.

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Proof (contd...)

For the backward implication in (ii), suppose that, for each $i \in \mathbb{N}_n, \pi_i \circ f$ is a Lipschitz function with Lipschitz constant ℓ_i.

Let $k = \sum_{i=1}^{n} \ell_i$. Then, for eac $a, b \in Z$, we have
$\tau_i(\pi_i(f(a)), \pi_i(f(b))) \leq \ell_i m(a, b)$, whence
$\sum_{i=1}^{n} \tau_i(\pi_i(f(a)), \pi_i(f(b))) \leq km(a, b)$.

Then, because e is a conserving metric, we get $e(f(a), f(b)) \leq km(a, b)$, as required.

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Exercises 91.

Which of the following continuous functions are uniformly continuous on the specified set? Justify your answers. Use any theorems you wish.

(a) $f(x) = x^{17}\sin x - e^x \cos\ 3x\ 3\ on[0,\pi]$,

(b) $f(x) = x^3 on[0,1]$,

(c) $f(x) = x^3 on(0,1)$,

(d) $f(x) = x^3 on\mathbb{R}$,

(e) $f(x) = \frac{1}{x^3} on(0,]$,

(f) $f(x) = \sin\frac{1}{x^2} on(0,1]$,

(g) $f(x) = x^2 \sin\frac{1}{x} on(0,1]$.

Hints: To decide (a) and (b), use Theorem 19.2. Parts (c), (e), (f) and (g) can be settled using Theorem 19.5. Theorem 19.4 can also be used to decide (e) and (f); compare Example 6. One needs to restore to the definition to handle (d).

Exercises 92.

Prove that each of the following functions is uniformly continuous on the indicated set by directly verifying the $\varepsilon - \delta$*property in Definition* 19.1.

(a) $f(x) = \frac{x}{x+1} on [0, 2]$,

(b) $f(x) = \frac{5x}{2x-1} on [1, \infty)$.

Solution

(a) Discussion. Let $\varepsilon > 0$. We want

$$|\frac{x}{x+1} - \frac{y}{y+1}| < \varepsilon \quad \text{or} \quad |\frac{x-y}{(x+1)(y+1)}| < \varepsilon$$

For $|x-y|$ small, $x, y, \in [0, 2]$. since $x + 1 \geq 1$ and $y + 1 \geq 1$ for $x, y \in [0, 2]$, it suffices to get $|x-y| < \varepsilon$. So we let $\delta = \varepsilon$.
Formal Proof : Let $\varepsilon > 0$ and let $\delta = \varepsilon$. Then $x, y \in [0, 2]$ and $|x-y| < \delta = \varepsilon$ imply

$$|f(x) - f(y)| = |\frac{x-y}{(x+1)(y+1)}| \leq |x-y| < \varepsilon.$$

(b) (a) Discussion. Let $\varepsilon > 0$. We want
$|g(x) - g(y)| = |\frac{5y-5x}{(2x-1)(2y-1)}| < \varepsilon$ for $|x-y|$ small, $x \geq 1, y \geq 1$. For $x, y \geq 1, 2x - 1 \geq 1$ and $2y - 1 \geq 1$, so it suffices to get $|5y - 5x| < \varepsilon$. So let $\delta = \frac{\varepsilon}{5}$. You should write out the formula proof.

Exercises 93.

which of the following continuous functions is uniformly continuous on the specified set? Justify your answers, using appropriate theorems or Exercise 19.4(a).

(a) $\tan \ x \ on[0, \frac{\pi}{4}]$,

(b) $\tan \ x \ on[0, \frac{\pi}{2})$,

(c) $\frac{1}{x}\sin^2 x on(0, \pi]$,

(d) $\frac{1}{x-3} on(0, 3)$,

(e) $\frac{1}{x-3} on(3, \infty)$,

(f) $\frac{1}{x-3} on(4, \infty)$.

(a)) $\tan x$ is uniformly continuous on $[0, \frac{\pi}{4}]$ by Theorem 19.2.

(b) $\tan x$ is not uniformly continuous on $[0, \frac{\pi}{2})$ by Exercise 19.4(a), since the function is not bounded on that set.

(c) Let $\tilde{h}$ be as in Example 9. Then$(\sin x)\tilde{h}(x)$ s a continuous extension of $(\frac{1}{x})\sin^2 x$ on $[0, \pi]$. Apply Theorem 19.5.

(e) $\frac{1}{x-3}$ is not uniformly continuous on $(3, 4)$ by Exercise 19.4(a), so it is not uniformly continuous on $(3, \infty)$ either.

(e) (f) Remark. It is easy to give an $\varepsilon - \delta$ proof that $\frac{1}{x-3}$ is uniformly continuous on $(4, \infty)$. It is even easier to apply Theorem 19..4

Exercises 94.

(a) *Let f be a continuous function on $[0, \infty)$. Prove that if f is uniformly continuous on $[k, \infty)$ for some k, then f is uniformly continuous on $[0, \infty)$.*

(b) *Use (a) and Exercise 19.6(b) to prove that $\sqrt{x}$ is uniformly continuous on $[0, \infty)$.*

(a) We are given that f is uniformly continuous on $[k, \infty)$, and f is uniformly continuous on $[0, k+1]$ by Theorem 19.2. Let $\varepsilon > 0$. There exist δ_1 and δ_2 so that

$$|x - y| < \delta_1, x, y \in [k, \infty) \quad \text{imply} \quad |f(x) - f(y)| < \varepsilon, (1)$$

$$|x - y| < \delta_2, x, y \in [0, k+1] \quad \text{imply} \quad |f(x) - f(y)| < \varepsilon. (2)$$

Let $\delta = \min\{1, \delta_1, \delta_2\}$ and show that

$$|x - y| < \delta, x, y \in [0, \infty) \quad \text{imply} \quad |f(x) - f(y)| < \varepsilon.$$

Exercises 95.

Let $f(x) = x \ \sin(\frac{1}{x})$ *for* $x \neq 0$ *and* $f(x) = 0$.

(a) *Observe that* f *is continuous on* $\mathbb{R}$; *see Exercises* 17.3(f) *and 417.9(c).*

(b) *why is* f *uniformly continuous on any bounded subset of* $\mathbb{R}$?

(c) *Is* f *uniformly continuous on* $\mathbb{R}$?

(c) This is tricky, but it turns out that f is uniformly continuous on $\mathbb{R}$. A simple modification of Exercise 19.7(a) shows that it suffices to show that f is uniformly continuous on $[1, \infty)$ and $(-\infty, -1]$. This can be done using Theorem 19.6. Note that we cannot apply Theorem 19.6 on $\mathbb{R}$ because f is not differentiable at $x = 0$; also f' is not bounded near $x = 0$.

Exercises 96.

Accept the fact that the function $\tilde{h}$ in Example 9 is continuous on $\mathbb{R}$; prove that it is uniformly continuous on $\mathbb{R}$.

As in the solution to Exercise 19.9(c), it suffices to show that $\tilde{h}$ is uniformly continuous on $[1.\infty)$ and $(-\infty, -1]$. Apply Theorem 19.6.

Part - 3

One of the important applications of uniform continuity concerns the integrability of continuous functions on closed intervals.

We proved the uniform continuity theorem (Theorem 26) which states that any continuous function from a compact subset K of a metric space X into a metric space Y is uniformly continuous on K.

Theorem 26 is important in the context of Riemann integration, where to show that every continuous function $f : [a, b] \to \mathbb{R}$ is Riemann integrable, **what we essentially use is the uniform continuity of f.**

Ross-137

Riemann Integrable Functions

Let f be a bounded function on $[a, b]$ and P be a partition of $[a, b]$. Let M_i and m_i be the supremum and infimum, respectively, of $f(x)$ on the subinterval $[x_{i-1}, x_i]$. The upper sum of f corresponding to the partition P is

$$U(f, P) = \sum_{i=1}^{n} M_i(x_i - x_{i-1})$$

and the lower sum of f corresponding to the partition P is

$$L(f, P) = \sum_{i=1}^{n} m_i(x_i - x_{i-1}).$$

Since $M_i \geq m_i$ for each i, we have $U(f, P) \geq L(f, P)$. Moreover, $\sup L(f, P)$ over all partitions P is less than or equal to $\inf U(f, P)$ – they are called the **lower and upper Riemann integrals of f over** $[a, b]$ respectively.

If the upper and lower Riemann integrals are equal, then the common value is called the **Riemann integral** of f over $[a, b]$ and we say that f is **Riemann integrable** on $[a, b]$. To show that the integral exists, it is sufficient to find, for any $\varepsilon > 0$, a partition P such that

$$U(f, P) - L(f, P) < \varepsilon.$$

Theorem 97.

*If $f(x)$ is a (**uniformly**) continuous function on the closed, bounded interval $[a, b]$, then f is Riemann integrable on $[a, b]$.*

Note that the above result is true for piecewise-continuous functions as well. Most functions that commonly occur in applications are continuous or piecewise-continuous functions. Piecewise-continuous function is the one having not more than a finite number of jump discontinuities on $[a, b]$.

Proof

Let $\varepsilon > 0$ be given.

Since f is uniformly continuous on $[a, b]$, there exists $\delta > 0$ be such that $|f(x) - f(y)| < \varepsilon/(b - a)$ whenever $x, y \in [a, b]$ and $|x - y| < \delta$. Now choose N_0 so that $(b - a)/N_0 < \delta$, and let P be the regular partition of $[a, b]$ into N_0 subintervals. Note that the regular partitions have the property that each partition interval is exactly the same size.

We will have $x_i - x_{i-1} < \delta$ for all i, so $M_i - m_i < \frac{\varepsilon}{b-a}$ for all i. Then

$$U(f, P) - L(f, P) = \sum_{i=1}^{n} (M_i - m_i) \frac{b - a}{N_0} < \sum_{i=1}^{N_0} \frac{\varepsilon}{b - a} \frac{b - a}{N_0} = \varepsilon.$$

So $U(f, P) - L(f, P) < \varepsilon$, providing the integrability of f.

Let (X, d) be a metric space. For $E \subseteq X$ and $x \in X$. let

$$d(x, E) := \inf\{d(x, y) : y \in E\}.$$

Exercise 98.

Let (X, d) be a metric space. For $x_0 \in X$, the map $x \longmapsto d(x, x_0)$ is Lipschitz function form X to $\mathbb{R}$ (with usual metric), with Lipschitz constant $k = 1$. [Note that for $x, y \in X$, $d(x, x_0) - d(y, x_0) \leq d(x, y), d(y, x_0) - d(x, x_0) \leq d(x, y)$. Hence, $|d(x, x_0) - d(y, x_0)| \leq d(x, y)$.]

Exercise 99.

Let (x_n) be a sequence in a metric space (X, d). Show that the function $x \longmapsto \inf\{d(x, x_n) : n \in \mathbb{N}\}$ is uniformly continuous on X. Kumaresan-77

Another useful criterion that implies uniform continuity

Here is another useful criterion that implies uniform continuity.

Theorem 100.

Let f be a continuous function on an interval I [I may be bounded or unbounded]. Let I° be the interval obtained by removing from I any endpoints that happen to be in I. If f is differentiable on I° and if f' is bounded on I°, then f is Lipschitz function on I.

Proof : For this proof we need the Mean Value Theorem. Let M be a bound for f' on I so that $|f'(x)| \leq M$ for all x. Consider $a, b \in I$ where $a < b$. By the Mean Value Theorem, there exists $x \in (a, b)$ such that $f'(x) = \frac{f(b)-f(a)}{b-a}$, so

$$|f(b) - f(a)| = |f'(x)| \cdot |b - a| \leq M|b - a|.$$

This proves the Lipschitz continuity of f on I. Ross-142

Another useful criterion that implies uniform continuity

Corollary 101.

Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable with $|f'(x)| \leq M$. Then f is Lipschitz.

Example 102.

Let $a > 0$ and consider $f(x) = \frac{1}{x^2}$. Since $f'(x) = -\frac{2}{x^3}$ we have $|f'(x)| \leq \frac{2}{a^3}$ on $[a, \infty)$. Hence f is Lipschitz function on $[a, \infty)$ by Theorem 100.

Ross-144, Kumaresan-78, Michael-156,157

Suppose (X, d) and (Y, ρ) are non-empty metric spaces, $z \in X$ and $f : X \to Y$. Rephrased, the epsilon-delta ball criterion for continuity of f at z is that, for each $\varepsilon > 0$, the set

$$S_{z,\varepsilon} = \{\delta > 0 \mid f(b[z;\delta)) \subseteq b[f(z);\varepsilon)\}$$

is not empty.

Once the existence of some δ in $S_{z,\varepsilon}$ is established, it is clear that $(0, \delta] \subseteq S_{z,\varepsilon}$. This means that every positive number smaller than δ would satisfy the requirement for continuity just as well as δ itself.

Is there a maximum value that δ can have? In other words, is $\sup S_{z,\varepsilon} \in S_{z,\varepsilon}$? The supremum can be infinite, in which case the answer to the question is no. But, if $x \in X$ and $d(x, z) < \sup S_{z,\varepsilon}$, then $d(x, z) < \delta$ for some $\delta \in S_{z,\varepsilon}$ (B.6.6), so that $e(f(x), f(z)) < \varepsilon$. It follows that if the supremum is finite, it is a member of $S_{z,\varepsilon}$.

A further question now arises. The function $\varepsilon \mapsto \sup S_{z,\varepsilon}$ is certainly decreasing in that if $\mu \in (0, \varepsilon)$, then $S_{z,\mu} \subseteq S_{z,\varepsilon}$, so that $\sup S_{z,\mu} \leq \sup S_{z,\varepsilon}$.

Can we say anything about the ratio of ε to $\sup S_{z,\varepsilon}$ as ε tends to 0? The question is put partly out of curiosity and partly because we think that, in some very special cases when f is a real differentiable function, there ought to be a relationship between the derivative and this ratio. We give some examples to illustrate that, even for real functions, all sorts of things can happen to the ratio.

- If f is constant, then $\sup S_{z,\varepsilon}$ is infinite for all ε and all $z \in X$.
- If f is not constant but is constant for all x sufficiently close to z, then $\sup S_{z,\varepsilon}$ is larger than some fixed positive number for all ε, so that $\varepsilon / sup\ S_{z,\varepsilon} \to 0$ as $\varepsilon \to 0$.

- If f is an isometry, then, saving exceptional cases, $\sup S_{z,\varepsilon} = \varepsilon$ for all ε, so that $\varepsilon / \sup S_{z,\varepsilon} \to 1$ as $\varepsilon \to 0$.
- If $X = [-1, 1]$ and $y = \mathbb{R}$ and $f(x) = \sqrt{1 - x^2}$ for all $x \in [-1, 1]$, then $\varepsilon / \sup S_{1,\varepsilon} \to \infty$ as $\varepsilon \to 0$. Note also that $|f'(x)| \to \infty$ as $x \to 1$ in$[-1, 1]$.
- If f is the modified step function defined on $[0, 1]$by

$$f(x) = \begin{cases} \frac{2^n+1}{2}x - \frac{2^n-1}{2^n}, & \text{if } n \in \mathbb{N} \text{ and } \frac{2}{2^n+1} \leq x \leq \frac{1}{2^{n-1}}, \\ \frac{1}{2^n}, & \text{if } n \in \mathbb{N} \text{ and } \frac{1}{2^n} < x < \frac{2}{2^n+1} \\ 0, & \text{if } x = 0, \end{cases}$$

then f is continuous at every point of [0,1] and differentiable at all points other than those where $x = 0$ or $x = 2/(2^n + 1)$ or $x = 1/2^n$ for some $n \in \mathbb{N}$.

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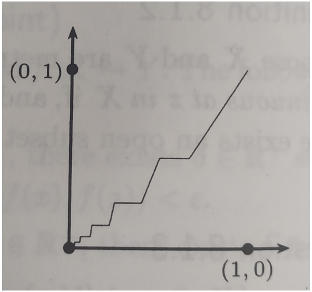
Local Continuity

The values of $\varepsilon / \sup S_{0,\varepsilon}$ range between $1/2$ and 1 as $\varepsilon \to 0$.

However, for each $n \in \mathbb{N}$ and $\varepsilon \in [0, 1/2^{n+1}]$, we have

$$\varepsilon / \sup S_{2^{-n}, \varepsilon} = 2^n + 1/2.$$



Differentiable Lipschitz Functions

Let us recall the ratio ε/δ that we discussed for continuous functions. For each $\varepsilon > 0$, a uniformly continuous function admits a corresponding $\delta > 0$, applicable now across the whole of the domain, that enables the function to satisfy the condition for uniform continuity. But we know that as smaller and smaller values are taken for ε, there is no guarantee that admissible values of δ follow a regular pattern.

For a Lipschitz function f, however, the ratio ε/δ need never exceed any Lipschitz constant for f. And our comment about differentiable functions is justified by the following theorem.

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Theorem 103.

Suppose I is a non-degenerate interval of $\mathbb{R}$ and $f : I \to \mathbb{R}$ is differentiable on I. then f is a Lipchitz function on I if, and only if, f' is bounded on I.

Proof : Suppose first that $k > 0$ and that $|f'(x)| \leq k$ for all $x \in I$. Suppose $a, b \in I$ and $a \neq b$. By the Mean Value Theorem, there exists $c \in I$ with c between a and b such that $f(b) - f(a) = (b - a)f'(c)$. This yields $|f(b) - f(a) \leq k|b - a|$. So f is a Lipschitz function with Lipchitz constant k.

For the converse, suppose that f' is not bounded on I and let $r > 0$ be arbitrary. Then there exist $a, b \in I$ such that $(f(b) - f(a))/(b - a) > r$, whence $|f(b) - f(a)|.r|b - a|$. So, since r is arbitrary positive real, f is not Lipschitz.

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Example 104.

The function $x \mapsto \sqrt{1-x^2}$ *defined on* $[0,1]$ *is differentiable on the interval* $[0,1)$*; in fact, the derivative is continuous.*
But the derivative is bounded only on intervals $[0,\alpha]$ *for* $\alpha \in (0,1)$*; it is not bounded on* $[0,1)$*. So this function is Lipschitz on every interval* $[0,\alpha]$ *with alpha* $\in (0,1)$ *but not Lipschitz on* $[0,1)$*. It is, however, uniformly continuous on* $[0,1]$ *simply because it is continuous on this closed bounded interval.*

Example 105.

The function x^2 *has bounded derivative* $2x$ *on every bounded interval of* $\mathbb{R}$*, so that, although* x^2 *is not even uniformly continuous on* $\mathbb{R}$*, it is Lipschitz on every bounded interval of* $\mathbb{R}$*.*

Linear Maps Between Normed Spaces

Linear maps between normed linear spaces have an extraordinary property that makes their continuity very much easier to handle than that of other maps: continuity at any one point of the domain implies Lipschitz continuity throughout the domain.

Theorem 106.

Let $T : X \to Y$ be a linear map between normed spaces X and Y. Then T is continuous iff it is Lipschitz on X.

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Theorem 107.

Let (X, d) be a complete metric space. If $f : X \to X$ is a contraction with contraction constant k $(0 < k < 1)$, then f has a **unique fixed point** *p, and for any x_0 in X the sequence $(f^n(x_0))$ converges to p.*

In fact,

$$d(f^n(x_0), p) \leq \frac{k^n}{1-k} d(x_0, f(x_0)). \qquad (\star)$$

Proof : Given that f is a contraction on X with contraction constant $k \in (0, 1)$. We have $d(f(x), f(y)) \leq k \; d(x, y)$ for all $x, y \in X$.

Let $x_0 \in X$ and let $(x_n)_{n \geq 1}$ be the sequence defined iteratively by $x_{n+1} = f(x_n)$ for $n = 0, 1, 2, \ldots$. We shall prove that $(x_n)_{n \geq 1}$ is a Cauchy sequence.

Existence

For $\ell = 1, 2, \ldots,$ we have

$$d(x_{\ell+1}, x_\ell) = d(f(x_\ell), f(x_{\ell-1})) \leq k \; d(x_\ell, x_{\ell-1}).$$

Repeated application of the above inequality gives

$$\begin{aligned} d(x_{\ell+1}, x_\ell) &\leq k \; d(x_\ell, x_{\ell-1}) \\ &\leq k^2 \; d(x_{\ell-1}, x_{\ell-2}) \leq \cdots \leq k^\ell \; d(x_1, x_0). \end{aligned}$$

Now, let n, p be positive integers. By the triangle inequality,

$$\begin{aligned} d(x_{n+p}, x_n) &\leq d(x_{n+p}, x_{n+p-1}) + d(x_{n+p-1}, x_{n+p-2}) + \cdots d(x_{n+1}, x_n) \\ &\leq \left[k^{n+p-1} + k^{n+p-2} + \cdots + k^n\right] \; d(x_1, x_0) \\ &\leq k^n \left[k^{p-1} + k^{p-2} + \cdots + 1\right] \; d(x_1, x_0) \\ &\leq k^n \left[\frac{1-k^p}{1-k}\right] \; d(x_1, x_0). \end{aligned}$$

Existence

For large values of n and p, we have k^n and k^p become smaller near 0 since $\lim_{n\to\infty} k^n = 0$. It follows that (x_n) is a Cauchy sequence in X, which is complete. Let $p = \lim_{n\to\infty} x_n$.

Since f is contraction, it is continuous. It follows that

$$f(p) = f(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_{n+1} = p.$$

Thus, p is a fixed point of f.

Uniqueness : Note that we cannot have more than one fixed point.

If there are two fixed points, say p, q, we would have $d(f(p), f(q)) = d(p, q)$. But we know that $d(f(p), f(q)) \leq k\ d(p, q)$ where $k < 1$ since f is a contraction.

Importanace of the Inequality ($\star$)

The importanace of the inequality ($\star$) (given in Banach contraction theorem) is as follows : Suppose we are willing to accept an "error" of ε, i.e., instead of the actual fixed point p of f we will be statisfied with a point p' of X satisfying $d(p, p') < \varepsilon$, and suppose also that we start our iteration at some point x_0 in X.

Then from the inequality it is easy to specify an integer N so that $p' = f^N(x_0)$ will be a satisfactory answer. Since we want $d(f^N(x_0), p) < \varepsilon$, we just have to pick N so large that $\frac{k^N}{1-k} d(x_0, f(x_0)) < \varepsilon$.

Now the quantity $D = d(x_0, f(x_0))$ is something that we can compute after the first iteration and we can then compute how large N has to be by taking the log of the inequality ($\star$) and solving for N (remembering that $\log(k)$ is negative).

The result is the **stopping rule.**

If $D = d(x_0, f(x_0))$ and

$$N > \frac{\log(\varepsilon) + \log(1-k) - \log D}{\log k}$$

then $d(f^N(x_0), p) < \varepsilon$.

Exercise 108.

Prove the inequality $(\star)$.

Exercises 109.

1. *Suppose X is a metric space and C is a non-empty closed subset of X. Show that $x \mapsto dist(x, C)$ is uniformly continuous on X.*
2. *Show that the function $f : x \mapsto x^2$ is uniformly continuous on the set $S = U\{[n, n + n^{-2}] | n \in \mathbb{N}\}$.*
3. *Show that not every product metric ensures the uniform continuity of the natural projections.*
4. *Give an example to show that the image of an open set under a uniformly continuous map need not be open. Is the same true for Lipschitz maps? For contractions? For isometric maps?*
5. *Suppose (X, d) and (Y, ρ) are metric spaces and $f : X \to Y$. Suppose there exists $k > 0$ such that $e(f(a), f(b)) \geq kd(a, b)$ for all $a, b \in X$. Show that f is injective and that f^{-1} is a Lipschitz function. Show also that f is an open mapping if $f(X)$ is open in Y.* *Michael-163*

Exercises 110.

1. *Suppose* X *is a non-empty set,* (Y, ρ) *is a metric space and* $S \subseteq B(X, Y)$. *For each* $x \in X$, *let* $\hat{x}$ *denote the function* $f \mapsto f(x)$ *defined on* S *(see 9.4.6). Show that* $\{\hat{x} | x \in X\}$ *is a bounded subset of* $e(S, Y)$ *if, and only if,* S *is bounded in* $B(X, Y)$.
2. *Suppose* X *and* Y *are normed linear spaces and* $f : X \to Y$ *is a linear map. Show that if* f *is a bounded function, then* $f = 0$.
3. *Suppose* (X, d) *and* (Y, ρ) *are metric spaces and* (f_n) *is a sequence of uniformly continuous functions from* X *to* Y *that converges uniformly to a function* $g : X \to Y$. *Show that* g *is uniformly continuous.*

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Exercises 111.

1. *Suppose* (X, d) *and* (Y, ρ) *are metric spaces and* (f_n) *is a sequence of functions from* X *to* Y *that are Lipschitz with Lipschitz constant* $k > 0$. *Suppose that* (f_n) *converges uniformly to* $g : X \to Y$. *Show that* g *is Lipschitz with Lipschitz constant* k.
2. *Show that the function* $x \mapsto x + x^{-1}$ *defined on* $[1, \infty)$ *is a contraction that is not strong.*

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