Double Integrals

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We defined the definite integral of a continuous function $f(x)$ over an interval $[a, b]$ as a limit of Riemann sums.

In the lecture we extend this idea to define the integral of a continuous function of two variables $f(x, y)$ over a bounded region $R$ in the plane.
In both cases the integrals are limits of approximating Riemann sums. The Riemann sums for the integral of a single-variable function $f(x)$ are obtained by partitioning a finite interval into thin subintervals, multiplying the width of each subinterval by the value of $f$ at a point $c_k$ inside that subinterval, and then adding together all the products.
A similar method of partitioning, multiplying, and summing is used to construct double integrals. However, this we pack a planar region $R$ with small rectangles, rather than small subintervals. We then take the product of each small rectangle’s area with the value of $f$ at a point inside that rectangle, and finally sum together all these products.

When $f$ is continuous, these sums converge to a single number as each of the small rectangles shrinks in both width and height. The limit is the double integral of $f$ over $R$. 
The major practical problem that arises in evaluating multiple integrals lies in determining the limits of integration.

While the integrals of single variable were evaluated over an interval, which is determined by its two endpoints, multiple integrals are evaluated over a region in the plane or in space.
We begin our investigation of double integrals by considering the simplest type of planar region, a rectangle. We consider a function $f(x, y)$ defined on a rectangular region $R$,

$$R : \quad a \leq x \leq b, \quad c \leq y \leq d.$$ 

The lines divide $R$ into $n$ rectangular pieces, where the number of such pieces $n$ gets large as the width and height of each piece gets small.
These rectangles form a **partition** of \( R \). A small rectangular piece of width \( \Delta x \) and height \( \Delta y \) has area

\[
\Delta A = \Delta x \Delta y.
\]

If we number the small pieces partitioning \( R \) in some order, then their areas are given by numbers

\[
\Delta A_1, \ \Delta A_2, \ldots, \ \Delta A_n
\]

where \( \Delta A_k \) is the area of the \( k \)th small rectangle.
To form a Riemann sum over $R$, we choose a point $(x_k, y_k)$ in the $k$th small rectangle, multiply the value of $f$ at that point by the area $\Delta A_k$, and add together the products

$$S_n = \sum_{k=1}^{n} f(x_k, y_k) \Delta A_k.$$ 

Depending on how we pick $(x_k, y_k)$ in the $k$th small rectangle, we may get different values for $S_n$. 
We are interested in what happens to these Riemann sums as the widths and heights of all the small rectangles in the partition of $R$ approach zero.

The **norm** of a partition $P$, written $\|P\|$, is the largest width or height of any rectangle in the partition.
 Sometimes the Riemann sums converge as the norm of $P$ goes to zero, written $\|P\| \to 0$. The resulting limit is then written as

$$\lim_{\|P\| \to 0} \sum_{k=1}^{n} f(x_k, y_k) \Delta A_k.$$ 

As $\|P\| \to 0$ and the rectangles get narrow and short, their number $n$ increases, so we can also write this limit as

$$\lim_{n \to \infty} \sum_{k=1}^{n} f(x_k, y_k) \Delta A_k$$

with the understanding that $\Delta A_k \to 0$ as $n \to \infty$ and $\|P\| \to 0$. 
There are many choices involved in a limit of this kind. The collection of small rectangles is determined by the grid of vertical and horizontal lines that determine a rectangular partition of $R$. In each of the resulting small rectangles there is a choice of an arbitrary point $(x_k, y_k)$ at which $f$ is evaluated. These choices together determine a single Riemann sum.

To form a limit, we repeat the whole process again and again, choosing partitions whose rectangle widths and heights both go to zero and whose number goes to infinity.
When a limit of the sums $S_n$ exists, giving the same limiting value no matter what choices are made, then the function $f$ is said to be integrable and the limit is called a **double integral** of $f$ over $R$, written as

$$\int\int_{R} f(x, y) \, dA \quad \text{or} \quad \int\int_{R} f(x, y) \, dx \, dy.$$  

It can be shown that if $f(x, y)$ is continuous throughout $R$, then $f$ is integrable as in the single-variable case discussed earlier.
Many discontinuous functions are also integrable, including functions which are discontinuous only on a finite number of points or smooth curves.
When \( f(x, y) \) is a positive function over a rectangular region \( R \) in the \( xy \)-plane, we may interpret the double integral of \( f \) over \( R \) as the volume of the 3-dimensional solid region over the \( xy \)-plane bounded below by \( R \) and above by the surface \( z = f(x, y) \).
Double Integrals as Volumes

Each term \( f(x_k, y_k) \Delta A_k \) in the sum \( S_n = \sum f(x_k, y_k) \Delta A_k \) is the volume of a vertical rectangular box that approximates the volume of the portion of the solid that stands directly above the base \( \Delta A_k \).

The sum \( S_n \) thus approximates what we want to call the total volume of the solid. We define this volume to be

\[
\text{Volume} = \lim_{n \to \infty} S_n = \iiint_R f(x, y) \, dA
\]

where \( \Delta A_k \to 0 \) as \( n \to \infty \).
Double Integrals as Volumes

The following figure shows Riemann sum approximations to the volume becoming more accurate as the number $n$ of boxes increases.

(a) $n = 16$

(b) $n = 64$

(c) $n = 256$
Suppose that we wish to calculate the volume under the plane 

\[ z = 4 - x - y \]

over the rectangular region

\[ R : 0 \leq x \leq 2, \quad 0 \leq y \leq 1 \]

in the xy-plane.
Fibini’s Theorem for Calculating Double Integrals

If we apply the method of slicing, with slices perpendicular to the \( x \)-axis, then the volume is

\[
\int_{x=0}^{x=2} A(x) \, dx
\]

where \( A(x) \) is the cross-sectional area at \( x \).

For each value of \( x \), we may calculate \( A(x) \) as the integral

\[
\int_{y=0}^{y=1} (4 - x - y) \, dy
\]

which is the area under the curve \( z = 4 - x - y \) in the plane of the cross-section at \( x \).
Fibini’s Theorem for Calculating Double Integrals

In calculating $A(x)$, $x$ is held fixed and the integration takes place with respect to $y$.

Hence the volume of the entire solid is

$$
\text{Volume} = \int_{x=0}^{x=2} A(x) \, dx = \int_{x=0}^{x=2} \int_{y=0}^{y=1} (4 - x - y) \, dx \, dy.
$$
The expression on the right, called an **iterated** or **repeated integral**, says that the volume is obtained by integrating $4 - x - y$ with respect to $y$ from $y = 0$ to $y = 1$, holding $x$ fixed, and then integrating the resulting expression in $x$ with respect to $x$ from $x = 0$ to $x = 2$.

The limits of integration 0 and 1 are associated with $y$, so they are placed on the integral closest to $dy$. The other limits of integration, 0 and 2, are associated with the variable $x$, so they are placed on the outside integral symbol that is paired with $dx$. 
Fibini’s Theorem for Calculating Double Integrals

What would have happened if we had calculated the volume by slicing with planes perpendicular to the $y$-axis?

$$z = 4 - x - y$$

$$A(y) = \int_{x=0}^{x=2} (4 - x - y) \, dx$$
As a function of $y$, the typical cross-sectional is shown above. Hence the volume of the entire solid is

$$
\text{Volume} = \int_{y=0}^{y=1} \int_{x=0}^{x=2} (4 - x - y) \, dy \, dx.
$$

Do both iterated integrals give the value of the double integral?

The answer is ‘yes’, since the double integral measures the volume of the same region as the two iterated integrals.
Fubini’s Theorem (First Form)

A theorem published in 1907 by Guido Fubini says that the double integral of any continuous function over a rectangle can be calculated as an iterated integral in either order of integration.

**Theorem**

If \( f(x, y) \) is continuous throughout the rectangular region \( R : a \leq x \leq b, \ c \leq y \leq d \), then

\[
\int\int_{R} f(x, y) \, dA = \int_{c}^{d} \int_{a}^{b} f(x, y) \, dx \, dy = \int_{a}^{b} \int_{c}^{d} f(x, y) \, dy \, dx.
\]
Fubini’s Theorem (First Form)

Fubini’s Theorem says that double integrals over rectangles can be calculated as iterated integrals. Thus, we can evaluate a double integral by integrating with respect to one variable at a time.

Fubini’s Theorem also says that we may calculate the double integral by integrating in either order, a genuine convenience. When we calculate a volume by slicing, we may use either planes perpendicular to the $x$-axis or planes perpendicular to the $y$-axis.
To define the double integral of a function $f(x, y)$ over a bounded, nonrectangular region $R$, we begin by covering $R$ with a grid of small rectangular cells whose union contains all points of $R$.

This time, however, we cannot exactly fill $R$ with a finite number of rectangles in the grid lie partly outside $R$. A partition of $R$ is formed by taking the rectangles that lie completely inside it, not using any that are either partly or completely outside.
For commonly arising regions, more and more of $R$ is included as the norm of a partition (the largest width or height of any rectangle used) approaches zero. Once we have a partition of $R$, we number the rectangles in some order from 1 to $n$ and let $\Delta A_k$ be the area of the $k$th rectangle. We then choose a point $(x_k, y_k)$ in the $k$th rectangle and form the Riemann sum

$$S_n = \sum_{k=1}^{n} f(c_k, y_k) \Delta A_k.$$
As the norm of the partition forming $S_n$ goes to zero, $||P|| \to 0$, the width and height of each enclosed rectangle goes to zero and their number goes to infinity.

If $f(x, y)$ is a continuous function, then these Riemann sums converge to a limiting value, not dependent on any of the choices we made. This limit is called the **double integral** of $f(x, y)$ over $R$:

$$\lim_{||P||\to0} \sum_{k=1}^{n} f(x_k, y_k) \Delta A_k = \int_{R} \int f(x, y) \, dA.$$
If \( f(x, y) \) is positive and continuous over \( R \) we define the volume of the solid generated between \( R \) and the surface \( z = f(x, y) \) to be

\[
\int \int_R f(x, y) \, dA.
\]
If $R$ is a region like the one shown in the $xy$-plane in the following figure, bounded “above” and “below” by the curves $y = g_2(x)$ and $y = g_1(x)$ and on the sides by the lines $x = a, x = b$, we may again calculate the volume by the method of slicing.
We first calculate the cross-sectional area

\[ A(x) = \int_{y=g_1(x)}^{y=g_2(x)} f(x, y) \, dy \]

and then integrate \( A(x) \) from \( x = a \) to \( x = b \) to get the volume as an iterated integral:

\[ V = \int_{a}^{b} A(x) \, dx = \int_{a}^{b} \int_{y=g_1(x)}^{y=g_2(x)} f(x, y) \, dy \, dx. \]
Similarly, if $R$ is a region like the one shown in the following figure, bounded by the curves $x = h_2(y)$ and $x = h_1(y)$ and the lines $y = c, \ y = d$, the volume calculated by slicing is given by the iterated integral.

$$\text{Volume} = \int_c^d \int_{x=h_1(y)}^{x=h_2(y)} f(x, y) \ dx \ dy.$$
Fubini’s Theorem (Stronger Form)

That the iterated integrals both give the volume that we defined to be the double integral of $f$ over $R$ is a consequence of the following stronger form of Fubini’s Theorem.

**Theorem**

Let $f(x, y)$ be continuous on a region $R$.

- If $R$ is defined by $a \leq x \leq b$, $g_1(x) \leq y \leq g_2(x)$, with $g_1$ and $g_2$ continuous on $[a, b]$, then

$$
\int \int_R f(x, y) \, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) \, dy \, dx.
$$

- If $R$ is defined by $c \leq y \leq d$, $h_1(y) \leq x \leq h_2(y)$, with $h_1$ and $h_2$ continuous on $[c, d]$, then

$$
\int \int_R f(x, y) \, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y) \, dx \, dy.
$$
Example

Find the volume of the prism whose base is the triangle in the $xy$-plane bounded by the $x$-axis and the lines $y = x$ and $x = 1$ and whose top lies in the plane

$$z = f(x, y) = 3 - x - y.$$
\[ V = \int_0^1 \int_0^x (3 - x - y) \, dy \, dx = \int_0^1 \int_y^1 (3 - x - y) \, dx \, dy. \]
Fubini’s Theorem (Stronger Form)

Although Fubini’s Theorem assures us that a double integral may be calculated as an iterated integral in either order of integration, the value of one integral may be easier to find than the value of the other.

Example

Calculate

$$\int \int_{R} \frac{\sin x}{x} \, dA$$

where $A$ is the triangle in the $xy$-plane bounded by the $x$-axis, the line $y = x$, and the line $x = 1$.

If we integrate first with respect to $y$ and then with respect to $x$, we get $-\cos(1) + 1$ as the answer.

$$\int_{0}^{1} \left( \int_{0}^{x} \frac{\sin x}{x} \, dy \right) \, dx = -\cos 1 + 1 \approx 0.46.$$
But if we reverse the order of integration and attempt to calculate, we run into a problem because

$$\int \frac{\sin x}{x} \, dx$$

cannot be expressed in terms of elementary functions (there is no simple antiderivative).

There is no general rule for predicting which order of integration will be good. If the order we first choose doesn’t work, we try the other. Sometimes neither order will work, and then we need to use numerical approximations.
Finding Limits of Integration

We now give a procedure for finding limits of integration that applies for many regions in the plane. Regions that are more complicated, and for which this procedure fails, can often be split up into pieces on which the procedure works.

Using Vertical Cross-sections

When faced with evaluation

\[ \int \int_R f(x, y) \, dA, \]

integrating first with respect to \( y \) and then with respect to \( x \), do the following.
Finding Limits of Integration

1. **Sketch**: Sketch the region of integration and label the bounding curves.

2. **y limits**: Imagine a vertical line $L$ cutting through $R$ in the direction of increasing $y$. Mark the $y$-values where $L$ enters and leaves. These are the $y$-limits of integration and are usually functions of $x$ (instead of constants).

3. **x limits**: Choose $x$-limits that include all the vertical lines through $R$. 
Finding Limits of Integration

Using Horizontal Cross-sections

To evaluate the same double integral as an iterated integral with the order of integration reversed, use horizontal lines instead of vertical lines in steps 2 and 3.

Double integrals of continuous functions over nonrectangular regions have the same algebraic properties as integrals over rectangular regions. These properties are useful in computations and applications.
Example

Write an equivalent integral for the integral

\[ \int_{0}^{2} \int_{x^2}^{2x} (4x + 2) \, dy \, dx \]

and write an equivalent integral with the order of integration reversed.
An equivalent integral with the order of integration reversed is

\[
\int_0^4 \int_{y/2}^{\sqrt{y}} (4x + 2) \, dx \, dy = 8
\]
Properties of Double Integrals

If \( f(x, y) \) and \( g(x, y) \) are continuous, then

- **Constant Multiple** :
  \[
  \int \int_R c f(x, y) \, dA = c \int \int_R f(x, y) \, dA, \quad \text{any number } c.
  \]

- **Sum and Difference** :
  \[
  \int \int_R \{ f(x, y) \pm g(x, y) \} \, dA = \int \int_R f(x, y) \, dA \pm \int \int_R g(x, y) \, dA.
  \]
Properties of Double Integrals

If $f(x, y)$ and $g(x, y)$ are continuous, then

- **Domination**: 
  - $\int\int_{R} f(x, y) \, dA \geq 0$ if $f(x, y) \geq 0$ on $A$.
  - $\int\int_{R} f(x, y) \, dA \geq \int\int_{R} g(x, y) \, dA$ if $f(x, y) \geq g(x, y)$ on $R$. 
Properties of Double Integrals

- Additivity: \[ \int\int_{R} f(x, y) \, dA = \int\int_{R_1} f(x, y) \, dA + \int\int_{R_2} f(x, y) \, dA \]

if \( R \) is the union of two nonoverlapping regions \( R_1 \) and \( R_2 \).
The idea behind these properties is that integrals behave like sums. If the function \( f(x, y) \) is replaced by its constant multiple \( cf(x, y) \), then a Riemann sum for \( f \)

\[ S_n = \sum_{k=1}^{n} f(x_k, y_k) \Delta A_k \]

is replaced by a Riemann sum for \( cf \)

\[ \sum_{k=1}^{n} cf(x_k, y_k) \Delta A_k = c \sum_{k=1}^{n} f(x_k, y_k) \Delta A_k = cS_n. \]
Taking limits as $n \to \infty$ shows that

$$c \lim_{n \to \infty} S_n = c \int\int_R f \ dA$$

and

$$\lim_{n \to \infty} cS_n = \int\int_R cf \ dA$$

are equal. It follows that the constant multiple property carries over from sums to double integrals.

The other properties are also easy to verify for Riemann sums, and carry over to double integrals for the same reason.
Example

Find the volume of the wedgelike solid that lies beneath the surface

\[ z = 16 - x^2 - y^2 \]

and above the region \( R \) bounded by the curve \( y = 2\sqrt{x} \), the line \( y = 4x - 2 \), and the \( x \)-axis.
Solution

The volume is

\[ V = \int_0^2 \int_{y^2/4}^{(y+2)/4} (16 - x^2 - y^2) \, dx \, dy = \frac{20803}{1680} \]
1. State Fubini’s Theorem for a continuous function on a region.

2. Write two properties of double integrals.

3. Evaluate

\[ \int_{-2}^{2} \int_{\pi/6}^{5\pi/6} \left\{ x^3 e^{\cos^2 y} + \sec \left( \frac{x}{2} \right) \cos y \right\} \, dy \, dx. \]

4. Sketch the region of integration, determine the order of integration, and evaluate the integral

(a) \[ \int \int_{R} (y - 2x^2) \, dA \text{ where } R \text{ is the region inside the square} \]

\[ |x| + |y| = 1. \]

(b) \[ \int_{0}^{3} \int_{\sqrt{x/3}}^{1} e^y^3 \, dy \, dx. \]
4. (a) \[ \int \int_{R} (y - 2x^2) \, dA = \int_{-1}^{0} \int_{x-1}^{x+1} (y - 2x^2) \, dy \, dx = \int_{0}^{1} \int_{x-1}^{1-x} (y - 2x^2) \, dy \, dx = -\frac{2}{3} \]

(b) \[ \int_{0}^{3} \int_{\sqrt{x/3}}^{1} e^{y^3} \, dy \, dx = \int_{0}^{1} \int_{0}^{3y^2} e^{y^3} \, dx \, dy = e - 1 \]
5. Using double integral find the volume of the wedge cut from the first octant by the cylinder $z = 12 - 3y^2$ and the plane $x + y = 2$.

6. Sketch the region of integration, write an equivalent double integral with order of integration reversed and evaluate the integral:

$$
\int_0^1 \int_y^1 x^2 e^{xy} \, dx \, dy.
$$

7. Evaluate the improper integral

$$
\int_1^{\infty} \int_{e^{-x}}^1 \frac{1}{x^3 y} \, dy \, dx.
$$
5. \[ V = \int_0^2 \int_0^{2-x} (12 - 3y^2) \, dy \, dx = 20 \]

6. \[ \int_0^1 \int_y^1 x^2 e^{xy} \, dx \, dy = \int_0^1 \int_0^x x^2 e^{xy} \, dy \, dx = \frac{e - 2}{2} \]

7. \[ \int_1^\infty \int_1^{e^{-x}} \frac{1}{x^3y} \, dy \, dx = \int_1^\infty \left[ \frac{\ln y}{x^3} \right]_1^{e^{-x}} \, dx = - \lim_{b \to \infty} \left[ \frac{1}{x} \right]_1^b = 1 \]
8. A symmetrical urn holds 24 buckets of water when it is full. The interior has a circular cross-section whose radius reduces from 3 m at the centre to 2 m at the base and top. The height (between base and top) of the urn is 12 m. The bounding surface of the urn is generated by revolving a parabola. When 6 buckets of water is stored, what would be the level of water (in the urn) from the bottom?
The equation of the parabola is \( y = -\left(\frac{z}{6}\right)^2 + 3 \). It is given that

\[
\int_{z=-6}^{z_0} \pi \left[ 3 - \left(\frac{z^2}{36}\right)^2 \right] \, dz = \frac{1}{4} \int_{z=-6}^{6} \pi \left[ 3 - \left(\frac{z^2}{36}\right)^2 \right] \, dz.
\]

We get the relation

\[
9z_0 - \frac{z_0^3}{18} + \frac{z_0^5}{5 \times 36^2} = -21.60.
\]

Hence the level of the water is \( z_0 + 6 \) m, where \( z_0 \) satisfies the above relation. Note that \( z_0 \) is negative and it is 3.51 m.
9. Set up the integral to find volume of the solid :
   (a) The wedge cut from the cylinder $x^2 + y^2 = 1$ by the planes $z = -y$ and $z = 0$.
   (b) The tetrahedron in the first octant bounded by the coordinate planes and the plane $x + \frac{y}{2} + \frac{z}{3} = 1$.

10. Let $D$ be the region in the first quadrant bounded by the curves $y = x$ and $y = x^3$. Integrate $h(x, y) = 6x^2e^{y^2}$ over $D$. 
11. Find the volume of the solid bounded by the surfaces $z = 3(x^2 + y^2)$ and $z = 4 - (x^2 + y^2)$.

12. Sketch the region of integration and evaluate the double integral

$$\int_{-1}^{1} \int_{-|x|}^{-1} \sin(y^2) \, dy \, dx.$$
References

3. N. Piskunov, Differential and Integral Calculus, Vol I & II (Translated by George Yankovsky).